On the Deficiency of Certain Surfaces.

By A. Cayley.

If a given point or curve is to be an ordinary or singular point or curve on a surface of the order n, this imposes on the surface a certain number of conditions, which number may be termed the „Postulation“; thus „Postulation of a given curve *quà* i-tuple curve on a surface n“ will denote the number of conditions to be satisfied by the surface in order that the given curve may be an i-tuple curve on the surface.

The „deficiency“ (Flächengeschlecht) of a given surface of the order n is

$$= \frac{1}{6}(n-1)(n-2)(n-3) \text{ less deficiency-value of the several singularities}$$

viz. as shown by Dr. Noether, if the surface has a given i-tuple curve, the deficiency-value hereof is

$$= \text{Postulation of the curve } \textit{quà } (i-1) \text{ tuple curve on a surface } n-4;$$

and if the surface has an i-conical point, the deficiency-value hereof is

$$= \text{Postulation of the point } \textit{quà } (i-2) \text{ conical point (on a surface } n-4\text{); viz. this is } = \frac{1}{6}i(i-1)(i-2),$$

and is thus independent of the order of the surface.

I remark that if the tangent-cone at the i-conical point has δ double lines and $\varkappa$ cuspidal lines, then the deficiency-value is

$$= \frac{1}{6}i(i-1)(i-2) + (i-2)(n-i-1)(\delta+\varkappa).$$

In the case of a double or cuspidal curve i is $= 2$, and the deficiency-value is

$$= \text{Postulation of given curve } \textit{quà} \text{ simple curve on a surface } n-4;$$

and so for an ordinary conical point i is $= 2$, and the deficiency-value is $= 0$: results which were first obtained by Dr. Clebsch.

I found in this manner the expression for the deficiency of a surface n having a double and cuspidal curve and the other singularities considered in my „Memoir on the theory of Reciprocal surfaces“ Phil. Trans. vol. 159 (1869); viz. this was

$$D = \frac{1}{6}(n-1)(n-2)(n-3) - (n-3)(b+c) + \frac{1}{2}(q+r) + 2t + \frac{1}{2}\beta + \frac{1}{2}\gamma + i - \frac{1}{2}\Theta,$$

where we have[1]

- b, order of double curve,
- q, class of Do.,
- c, order of cuspidal curve,
- r, class of Do.,
- β, number of intersections of the two curves, stationary points on b,
- γ, number of intersections, stationary points on c,
- i, number of intersections, not stationary points on either curve,
- Θ, number of certain singular points on c, the nature of wich I do not completely understand, and which is here taken to be $=0$.

Before going further I remark that

Postulation of right line *quà* i-tuple on surface n

$$= \frac{1}{2}i(i+1)n - \frac{1}{6}i(i+1)(2i-5),$$
$$= \frac{1}{6}i(i+1)(3n-2i+5).$$

Whence if a surface n has a i-tuple right line, the deficiency-value hereof is

$$= \frac{1}{6}i(i-1)(3n-2i-5),$$

or we have

$$\begin{aligned} D &= \frac{1}{6}(n-1)(n-2)(n-3) - \frac{1}{6}i(i-1)(3n-2i-5) \\ &= \frac{1}{6}(i-n+1)(i-n+2)(2i+n-3); \end{aligned}$$

so that $D=0$ if either $i=n-1$ or $i=n-2$; the former case is that of a scroll (skew surface) with a $(n-1)$ tuple right line the latter that of a surface with a $(n-2)$ tuple line: whence (as shown by Dr. Noether) such surface is rationally transformable into a plane.

For a surface of the order n with a i-conical point where the tangent cone has δ double lines and $\varkappa$ cuspidal lines, we have

$$\begin{aligned} D &= \frac{1}{6}(n-1)(n-2)(n-3) - \{\frac{1}{6}i(i-1)(i-2) + (i-2)(n-i-1)(\delta+\varkappa)\} \\ &= \frac{1}{6}(n-i-1)\{n^2 + n(i-5) + i^2 - 4i + 6 - 6(i-2)(\delta+\varkappa)\} \end{aligned}$$

viz. for $i=n-1$ this is $D=0$ (in fact a surface n with a $(n-1)$ conical point is at once seen to be rationally transformable into a plane): and for $i=n$, that

[1] *Editorial note:* In the explanation of Θ in the list below, the printed text reads "the nature of wich I do not completely understand": so printed, without an "h", for "which".

is for a cone of the order n, we have

$$D = -\frac{1}{2}(n-1)(n-2) + (n-2)(\delta+\varkappa) - (n-3)(\delta+\varkappa),$$

where the last term $-(n-3)(\delta+\varkappa)$ is added because in the present case the surface has the δ double lines and the $\varkappa$ cuspidal lines.

The formula therefore gives

$$D = -\frac{1}{2}(n-1)(n-2) + \delta + \varkappa,$$

viz. this is equal to the deficiency of the plane sections *taken negatively*.

I find that the same property exists *first* in the case of a scroll (skew surface) having only a double curve; and *secondly* in the case of a torse (developable surface) having a cuspidal curve with the ordinary singularities; and this being so there can I think be no doubt but that it is true for any scroll or torse whatever — viz. that for any ruled surface whatever the deficiency is equal to that of the plane section taken negatively.

First for the scroll we have

$$D = \frac{1}{6}(n-1)(n-2)(n-3) - (n-3)b + \frac{1}{2}q + 2t,$$

which should be

$$= -\frac{1}{2}(n-1)(n-2) + b.$$

Salmon's equations give in the case of a scroll

$$\begin{aligned} 3t &= (n-4)\{3b - n(n-2)\}, \\ q &= n(n-2)(n-5) - 2(n-6)b, \end{aligned}$$

and with these values the relation is at once verified.

Secondly for the torse; changing the notation into that used for the singularities of the curve and torse, we have

$$D = \frac{1}{6}(r-1)(r-2)(r-3) - (r-3)(x+m) + \frac{1}{2}(q+r) + 2t + \frac{7}{2}\beta + \frac{5}{2}\gamma + \alpha,$$

which should be

$$= -\frac{1}{2}(m-1)(m-2) + h + \beta;$$

we have $q = r(n-3) - 3\alpha$, and substituting this value and expressing every thing in terms of r, m, n by means of the formula[2]

$$\begin{aligned}
x &= \frac{1}{2}(r^2 - r - n - 3m),\\
\alpha &= m - 3r + 3n,\\
\beta &= n - 3r + 3m,\\
t &= \frac{1}{6}\{r^3 - 3r^2 - 58r - 3r(n + 3m) + 42n + 78m\},\\
\gamma &= rm + 12r - 14m - 6n,\\
h &= \frac{1}{2}(m^2 - 10m - -3n + 8r),
\end{aligned}$$

we have after all reductions[3]

$$D = -\frac{1}{2}(m + n) + r - 1 = -\frac{1}{2}(m - 1)(m - 2) + h + \beta.$$

We have thus a class of surfaces of *negative deficiency*; viz. any rational transformation of a cone for which the plane section has a given (positive) deficiency produces such a surface: and I think it may be assumed conversely that a surface of negative deficiency is always the rational transformation of a cone for which the deficiency is equal to that of the surface taken with the reverse sign. As an instance take a quintic surface having a nodal conic and two 3-conical (cubiconical) points (this of course implies that the line joining the two cubiconical points is a line on the surface): the formula for the

$$D = \frac{1}{6}(n - 1)(n - 2)(n - 3) - (n - 3)b + \frac{1}{2}q + 2t - 2$$

(viz. a term -1 for each of the cubiconical points)

$$= 4 - 4 + 1 - 2, \quad = -1.$$

Such a surface can be obtained as the quadric inverse of a cubic cone; viz. taking for the vertex the point $x : y : z : \omega = \alpha : \beta : \gamma : \delta$ and the cone to pass thro' the point $x = 0, y = 0, z = 0$, the equation of the cone is[4]

$$(\delta x - \alpha\omega,\ \delta y - -\beta\omega,\ \delta z - \gamma\omega)^3 = 0$$

where $(\alpha, \beta, \gamma)^3 = 0$.

[2] *Editorial note:* In the expression for h in the following system, the print sets a double hyphen between $10m$ and $3n$, $10m - -3n$: reproduced here as printed.

[3] *Editorial note:* In the rightmost member of the following display, the minus sign in $(m-1)$ shows a small ink defect above the rule in the print, resembling a division sign at first glance; it is a defective minus sign, matching the identical factor $(m - 1)$ earlier on the same page.

[4] *Editorial note:* In the following equation the print sets a double hyphen between δy and $\beta\omega$, $\delta y - -\beta\omega$, matching the typographical slip in the formula for h on the preceding page: reproduced here as printed.

Taking Q a quadric function $(x, y, z)^2$, the transformation in question consists in the change of x, y, z, ω into $x\omega, y\omega, z\omega, Q$; viz. the new equation, rejecting the factor ω which divides out, is

$$\frac{1}{\omega}(\delta x\omega - \alpha Q,\ \delta y\omega - \beta Q,\ \delta z\omega - \gamma Q)^3 = 0$$

which is a quintic surface having the two cubiconical points $x = 0, y = 0, z = 0$ and $x : y : z : \omega = \alpha : \beta : \gamma : \frac{1}{\delta}Q_0$ (where Q_0 is the value of Q on writing therein α, β, γ in place of x, y, z): and having the nodal conic $\omega = 0, Q = 0$.

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