De integratione aequationis differentialis

$$\frac{m\,dx}{\sqrt{1-x^4}} = \frac{n\,dy}{\sqrt{1-y^4}}$$

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§. 1. Cum primum occasione inuentionum Ill. Comitis Fagnani hanc aequationem essem contemplatus, eiusmodi quidem relationem algebraicam inter variabiles x et y elicui, quae huic aequationi satisfaceret; sed ea relatio non pro aequatione integrali completa haberi poterat, propterea quod non complecteretur quantitatem constantem arbitrariam, cuiusmodi semper in calculum per integrationem introduci solet. Hinc enim, vti satis notum est, integralia incompleta et particularia distingui solent, quorum illa totam vim aequationum differentialium exhauriunt, haec vero tantum ita satisfaciunt, vt aliae insuper expressiones aeque satisfacere queant. Criterium autem aequationis integralis completae in hoc consistit, quod ea quantitatem constantem inuoluere debeat, quae in aequatione differentiali non apparet.

§. 2. Quae quo clarius perspiciantur, sufficiet, aequationem differentialem simplicissimam $dx = dy$ considerasse, cui vtique satisfacit haec integralis $x = y$, in rem tamen haec integralis minus late patet, quam differentialis $dx = dy$, cum huic aeque satisfaciat haec integralis $x = y + a$ multo latius patens, sumendo pro a quantitatem constantem quamcunque, atque haec demum integralis totam vim aequationis differentialis $dx = dy$ exhaurire censetur, ex quo etiam aequatio integralis completa appellatur; propterea quod in ea inest quantitas constans a, quae in aequatione differentiali non occurrit. Quodsi vero loco istius constantis indefinitae a valores determinati substituantur, ex integrali completo obtinentur integralia particularia, quae ob hanc ipsam rationem minus late patent, quam aequatio differentialis proposita.

§. 3. Saepe numero autem aequationis differentialis integrale particulare algebraicum exhiberi potest, cum tamen integrale completum sit transcendens; hoc scilicet euenit, si pars transcendens per constantem illam arbitrariam fuerit multiplicata, quae propterea, constante illa nihilo aequali posita, ex calculo euanescit, et integrale algebraicum particulare relinquit. Ita huic aequationi $dy = dx + (y-x)\,dx$ manifestum est, satisfacere valorem $y = x$; quo tamen tantum integrale particulare continetur, cum completum sit $y = x + ae^x$, denotante e numerum, cuius logarithmus est $= 1$. Nisi igitur constans arbitraria a euanescens ponatur, integrale semper erit transcendens.

§. 4. Cum igitur euenire queat, vt aequatio differentialis integrale particulare algebraicum admittat, etiamsi integrale completum sit transcendens, ita etiam rationes dubitandi non desunt, quod integrale completum aequationis differentialis propositae $\frac{m\,dx}{\sqrt{1-x^4}} = \frac{n\,dy}{\sqrt{1-y^4}}$ quantitates transcendentes inuoluat, etiamsi pro ea integrale particulare algebraicum exhibere licuerit. Cum enim integrale completum sit:[1]

$$m\int\frac{dx}{\sqrt{1-x^4}}=n\int\frac{dy}{\sqrt{1+y^4}}+C$$

haec autem integralia nullo modo, neque circuli, neque hyperbolae, quadraturam in subsidium vocando, assignari queant, minime probabile videtur, istas formulas tantopere transcendentes in genere, ita vt constans C maneat indeterminata, ad relationem algebraicam inter x et y reuocari posse.

§. 5. Notum quidem est, integrale completum huius aequationis differentialis $\frac{m\,dx}{\sqrt{1-xx}}=\frac{n\,dy}{\sqrt{1-yy}}$ semper algebraice exhiberi posse, dummodo proportio coefficientium m et n fuerit rationalis; sed quia vtriusque formulae integrale arcum circuli indicat, ita vt integrale completum sit $m\,\mathrm{A}\sin.x=n\,\mathrm{A}\sin.y+C$, relatio autem sinuum, qui ad arcus proportionem rationalem inter se tenentes spectant, algebraice exprimi potest, mirum non est, aequationem integralem completam his casibus quoque algebraice exhiberi posse. Cum autem huiusmodi comparatio in formulis transcendentibus $\int\frac{dx}{\sqrt{1-x^4}}$ et $\int\frac{dy}{\sqrt{1-y^4}}$ locum non habeat, seu saltem non constet, inde reductio integralis ad quantitates algebraicas peti non poterit.

§. 6. Nihilo tamen minus obseruaui, si proposita fuerit huiusmodi aequatio differentialis

$$\frac{m\,dx}{\sqrt{1-x^4}}=\frac{n\,dy}{\sqrt{1-y^4}}$$

etiam integrale completum, quod scilicet quantitatem constantem arbitrariam inuoluat, semper algebraice exprimi posse, dummodo ratio $m:n$ fuerit rationalis: quod mihi quidem eo magis notatu dignum videtur, quod nulla certa methodo ad hoc integrale sum perductus, sed id potius tentando, vel diuinando, elicui. Vnde nullum est dubium, quin methodus directa, ad idem hoc integrale perducens, fines analyseos non mediocriter sit amplificatura; cuius propterea inuestigatio Analystis omni studio commendanda videtur.

§. 7. Completum autem integrale aequationis istius differentialis, quaecunque fuerit ratio rationalis coëfficientium m et n, deriuare mihi licuit ex integratione completa huius aequationis $\frac{dx}{\sqrt{1-x^4}}=\frac{dy}{\sqrt{1-y^4}}$: hac enim concessa methodum certam indicabo, ex ea quoque integrale completum huius aequationis multo latius patentis $\frac{m\,dx}{\sqrt{1-x^4}}=\frac{n\,dy}{\sqrt{1-y^4}}$ concludendi. Quae methodus etiam in genere ad huiusmodi aequationum $mX\,dx=nY\,dy$ integralia inuenienda adhiberi queat, si modo integrale completum huius $X\,dx=Y\,dy$ fuerit erutum, atque Y talem significet functionem ipsius y, qualis X est ipsius x.

[1] *Editorial note:* In the printed source the second radical of this display is set with a plus, $\sqrt{1+y^4}$; an apparent misprint for $\sqrt{1-y^4}$ — the sign is negative in the differential equation itself and everywhere else. Reproduced here exactly as printed.

§. 8. Exordiar igitur ab hac aequatione

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$$

cui quidem primo intuitu satisfacere perspicuum est aequationem $x = y$, quae propterea eius est integrale particulare. Tum vero eidem aequationi quoque satisfacit iste valor algebraicus $x = -\sqrt{\frac{1-yy}{1+yy}}$, cum enim sit $dx = +\frac{2y\,dy}{(1+yy)\sqrt{(1-yy)(1+yy)}}$ et $\sqrt{1-x^4} = \frac{2y}{1+yy}$, erit $\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$. Hinc iste etiam valor, seu aequatio $xxyy+xx+yy-1 = 0$ est integralis particularis aequationis differentialis propositae. Vnde integrale completum, quod constantem arbitrariam inuoluat, ita comparatum sit necesse est, vt tribuendo huic constanti certum quendam valorem, prodeat $x = y$; sin autem eidem constanti alius quidem valor tribuatur, vt prodeat $x = -\sqrt{\frac{1-yy}{1+yy}}$ seu $xxyy + xx + yy - 1 = 0$.

Theorema.

§. 9. Dico igitur huius aequationis differentialis

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$$

aequationem integralem completam esse:

$$xx + yy + ccxxyy = cc + 2xy\sqrt{1-c^4}$$

Demonstratio.

Posita enim hac aequatione, eius differentiale erit:

$$x\,dx + y\,dy + ccxy(x\,dy + y\,dx) = (x\,dy + y\,dx)\sqrt{1-c^4}$$

vnde fit

$$dx\big(x + ccxyy - y\sqrt{1-c^4}\big) + dy\big(y + ccxxy - x\sqrt{1-c^4}\big) = 0$$

Ex eadem vero aequatione resoluta colligitur:

$$y = \frac{x\sqrt{1-c^4} + c\sqrt{1-x^4}}{1+ccxx} \quad \text{et} \quad x = \frac{y\sqrt{1-c^4} - c\sqrt{1-y^4}}{1+ccyy}$$

Si enim ibi radicali $\sqrt{1-x^4}$ tribuitur signum $+$, hic radicali $\sqrt{1-y^4}$ signum $-$ tribui debet, vt posito $x = 0$, vtrinque idem valor prodeat $y = c$. Erit ergo

$$x + ccxyy - y\sqrt{1-c^4} = -c\sqrt{1-y^4}$$

$$y + ccxxy - x\sqrt{1-c^4} = c\sqrt{1-x^4}$$

quibus valoribus in aequatione differentiali substitutis, prodit

$$-c\,dx\sqrt{1-y^4} + c\,dy\sqrt{1-x^4} = 0,$$

siue

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}.$$

Huius ergo aequationis differentialis integrale est:

$$xx + yy + ccxxyy = cc + 2xy\sqrt{1-c^4}$$

et quia constantem c ab arbitrio nostro pendentem continet, erit simul integrale completum. Q. E. D.

§. 10. Si igitur habeatur haec aequatio $\dfrac{dx}{\sqrt{1-x^4}} = \dfrac{dy}{\sqrt{1-y^4}}$ valor integralis completus ipsius x est:

$$x = \frac{y\sqrt{1-c^4} + c\sqrt{1-y^4}}{1+ccyy}$$

vnde si constans arbitraria c euanescat fit $x = y$; sin autem ponatur $c = 1$, habemus $x = \pm\dfrac{\sqrt{1-y^4}}{1+yy} = \sqrt{\dfrac{1-yy}{1+yy}}$ qui sunt ambo illi valores particulares iam supra exhibiti.

Hinc eruuntur alii valores particulares prae caeteris simpliciores, sed qui ad imaginaria deuoluuntur. Ita posito $c = \infty$ fit $x = \dfrac{\sqrt{-1}}{y}$; et posito $cc = -1$; fit $x = \sqrt{\dfrac{yy+1}{yy-1}}$ qui itidem aequationi propositae satisfaciunt.

§. 11. Quo autem ratio huius integralis clarius perspiciatur, concipiatur curua AM cuius haec sit indoles, vt posita abscissa $AP = u$, sit arcus ei respondens $AM = \displaystyle\int \frac{du}{\sqrt{1-u^4}}$. Deinde eadem curua denuo descripta, capiatur abscissa $ap = x$, erit arcus $am = \displaystyle\int \frac{dx}{\sqrt{1-x^4}}$. Sumto igitur

$$x = \frac{u\sqrt{1-c^4} + c\sqrt{1-u^4}}{1+ccuu}$$

fiet $\dfrac{dx}{\sqrt{1-x^4}} = \dfrac{du}{\sqrt{1-u^4}}$; ideoque arc. $am =$ arc. $AM +$ Const. Pro constantis autem huius determinatione, posito $u = 0$ quo casu arcus AM euanescit, fit $x = c$. Quare si capiatur abscissa $ab = c$, cui arcus ad respondeat, erit arcus $dm =$ arcui AM.

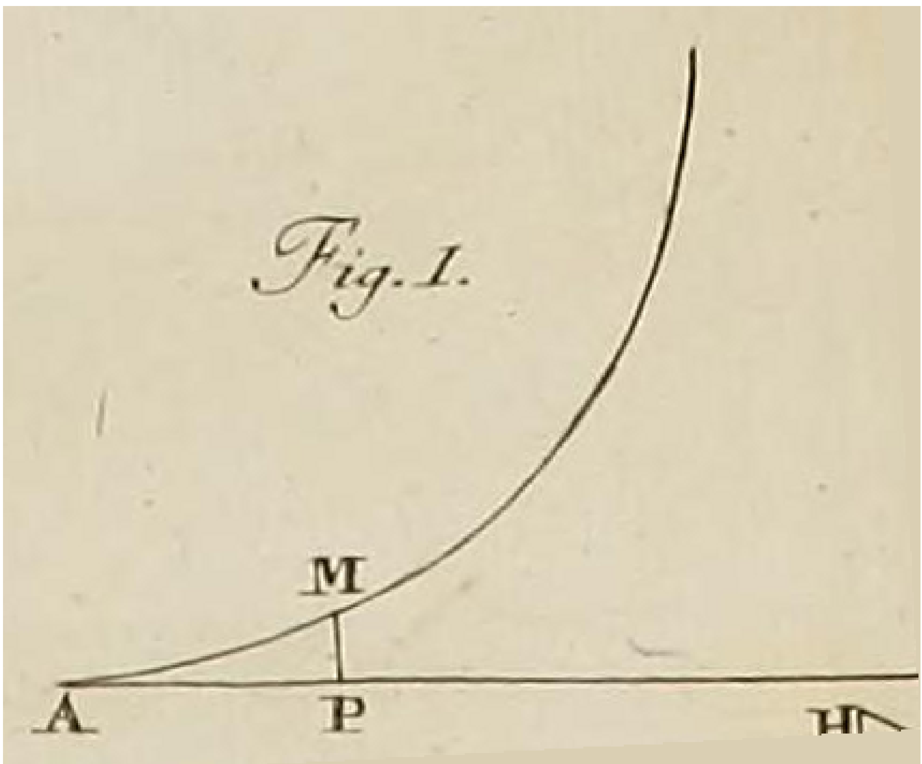

Figure 1: Fig. 1.

§. 12. Ope huius ergo integrationis completa aequationis $\frac{dx}{\sqrt{1-x^4}} = \frac{du}{\sqrt{1-u^4}}$, in curua proposita arcui cuicunque AM, qui abscissae $AP = u$ respondet, arcus aequalis dm, qui a dato puncto d incipiat, abscindi poterit. Posita enim abscissa dato puncto d respondente $ab = c$; si capiatur abscissa $ap = x = \frac{c\sqrt{1-u^4}+u\sqrt{1-c^4}}{1+ccuu}$ erit arcus dm arcui AM aequalis. Simili autem modo

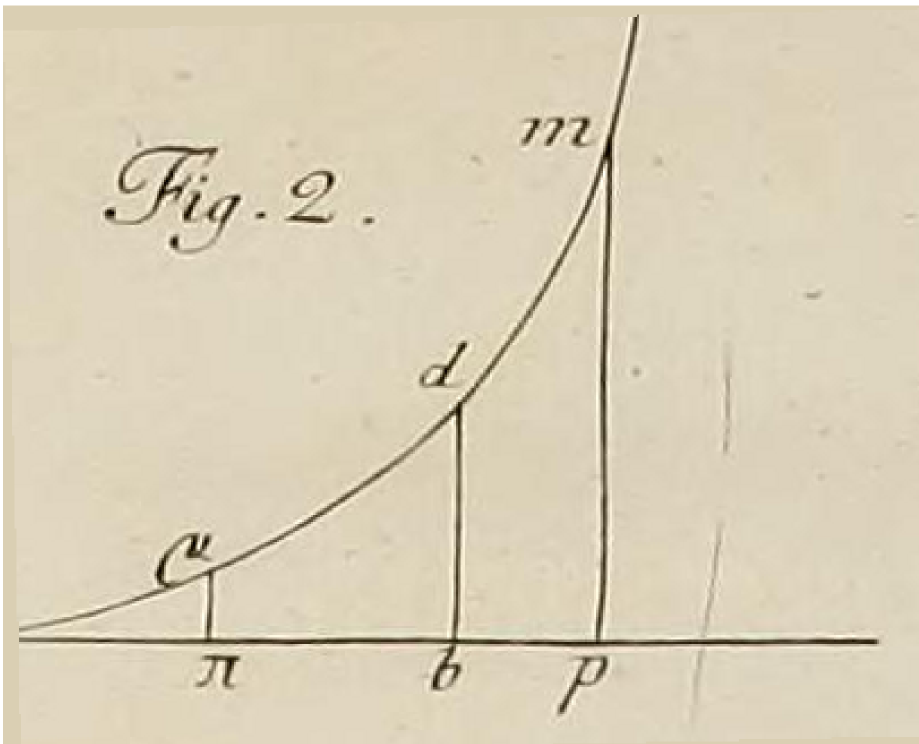

Figure 2: Fig. 2.

cum $\sqrt{1-c^4}$ negatiuum statui liceat, si capiatur abscissa

$$a\pi = \frac{c\sqrt{1-u^4}-u\sqrt{1-c^4}}{1+ccuu}$$

erit itidem arcus $d\mu$ arcui AM aequalis: sicque in hac curua a dato quouis puncto d vtrinque abscindi potest arcus dm et $d\mu$, qui arcui AM sint aequales.

§. 13. Hinc ergo patet, si arcus ad aequalis capiatur arcui AM, seu $c = u$, fore arcum am duplum arcus AM. Hinc si statuatur $ap = x = \frac{2u\sqrt{1-u^4}}{1+u^4}$, prodibit arcus $am = 2$ arc. AM. Simili modo si capiatur arcus $ad = 2AM$, seu $c = \frac{2u\sqrt{1-u^4}}{1+u^4}$, statuaturque $x = \frac{c\sqrt{1-u^4}+u\sqrt{1-c^4}}{1+ccuu}$ obtinebitur arcus

$am = 3$ arc. AM. Ac si iste valor ipsius x denuo pro c substituatur, vt fit $ad = 3AM$ iterumque statuatur $x = \dfrac{c\sqrt{1-u^4} + u\sqrt{1-c^4}}{1+ccuu}$, nascetur arcus am quadruplus arcus AM; atque ita porro successiue quaecunque multipla arcus AM geometrice assignari poterunt.

§. 14. Sit arcus $ad = n \cdot AM$ et $ab = z$; ita vt sit $\int \dfrac{dz}{\sqrt{1-z^4}} = n \int \dfrac{du}{\sqrt{1-u^4}}$; atque ex his patet si capiatur $x = \dfrac{z\sqrt{1-u^4} + u\sqrt{1-z^4}}{1+uuzz}$ fore $\int \dfrac{dx}{\sqrt{1-x^4}} = (n+1) \int \dfrac{du}{\sqrt{1-u^4}}$; sin autem ponatur $x = \dfrac{z\sqrt{1-u^4} - u\sqrt{1-z^4}}{1+uuzz}$, tum futurum esse $\int \dfrac{dx}{\sqrt{1-x^4}} = (n-1) \int \dfrac{du}{\sqrt{1-u^4}}$. Si igitur haec aequatio $\dfrac{dz}{\sqrt{1-z^4}} = \dfrac{n\,du}{\sqrt{1-u^4}}$ fuerit integrata, debitusque valor pro z inde erutus, etiam integrari poterit haec aequatio $\dfrac{dx}{\sqrt{1-x^4}} = \dfrac{(n \pm 1)\,du}{\sqrt{1-u^4}}$, quippe cuius integrale erit $x = \dfrac{z\sqrt{1-u^4} \pm u\sqrt{1-z^4}}{1+uuzz}$. Ac si pro z assumtus fuerit eius valor completus, qui scilicet constantem arbitrariam inuoluat, etiam pro x prodibit eius valor completus.

§. 15. Hinc igitur perspicuum est, quomodo aequatio integralis completa inueniri debeat, quae conueniat huic aequationi differentiali $\dfrac{dx}{\sqrt{1-x^4}} = \dfrac{n\,du}{\sqrt{1-u^4}}$; quoties n fuerit numerus integer. Simili autem modo assignari poterit y, vt fit $\dfrac{dy}{\sqrt{1-y^4}} = \dfrac{m\,du}{\sqrt{1-u^4}}$, vnde si eliminando u, aequatio inter x et y quaeratur, ea erit integralis huius aequationis $\dfrac{m\,dx}{\sqrt{1-x^4}} = \dfrac{n\,dy}{\sqrt{1-y^4}}$, quicunque numeri rationales pro m et n substituantur: atque vt hoc integrale prodeat completum, sufficit pro altera tantum variabilium x et y valorem completum per u determinasse, cum hinc iam noua constans arbitraria in calculum introducatur.

§. 16. Methodus, qua hic in Theorematis demonstratione sum vsus, etsi non ex rei natura est petita, sed indirecte ad id, quod propositum erat, perduxit, tamen multo latius patet: simili enim modo colligitur, huius aequationis differentialis

$$\frac{dx}{\sqrt{1+mxx+nx^4}} = \frac{dy}{\sqrt{1+myy+ny^4}}$$

integrale completum esse:

$$0 = cc - xx - yy + nccxxyy + 2xy\sqrt{1+mcc+nc^4}$$

Vnde idem, quod ante, ratiocinium adhibendo, integrale quoque completum obtinebitur huius aequationis

$$\frac{\mu\,dx}{\sqrt{1+mxx+nx^4}} = \frac{\nu\,dy}{\sqrt{1+myy+ny^4}}$$

siquidem litteris μ et ν numeri integri designentur.

§. 17. Inuestigatio autem huius integrationis ita se habet: Fingatur primo pro arbitrio relatio inter variabiles x et y hac aequatione contenta:

$$\alpha xx + \alpha yy = 2\beta xy + \gamma xxyy + \delta \tag{1}$$

quae differentiata dat:

$$\alpha x\, dx + \alpha y\, dy = \beta x\, dy + \beta y\, dx + \gamma xyy\, dx + \gamma xxy\, dy$$

vnde conficitur

$$dx(\alpha x - \beta y - \gamma xyy) + dy(\alpha y - \beta x - \gamma xxy) = 0 \tag{2}$$

Deinde ex aequatione (1) eliciantur valores vtriusque variabilis:

$$x = \frac{\beta y + \sqrt{\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4}}{\alpha - \gamma yy}$$

$$y = \frac{\beta x - \sqrt{\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4}}{\alpha - \gamma xx}$$

Atque hinc obtinemus:

$$\alpha x - \beta y - \gamma xyy = \sqrt{\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4} \tag{3}$$

$$\alpha y - \beta x - \gamma xxy = -\sqrt{\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4} \tag{4}$$

qui valores in aequatione (2) substituti praebebunt

$$\frac{dx}{\sqrt{\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4}} = \frac{dy}{\sqrt{\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4}} \tag{5}$$

cuius ergo aequationis integrale est aequatio (1).

§. 18. Quo istas formas simpliciores reddamus, ponamus $\alpha\delta = A$; $\beta\beta - \alpha\alpha - \gamma\delta = C$; $\alpha\gamma = E$ eritque $\delta = \dfrac{A}{\alpha}$; $\gamma = \dfrac{E}{\alpha}$ et $\beta = \sqrt{C + \alpha\alpha + \dfrac{AE}{\alpha\alpha}}$. Quare huius aequationis differentialis

$$\frac{dx}{\sqrt{A + Cxx + Ex^4}} = \frac{dy}{\sqrt{A + Cyy + Ey^4}} \tag{6}$$

aequatio integralis est haec:

$$\alpha(xx + yy) = \frac{A}{\alpha} + \frac{E}{\alpha}xxyy + 2xy\sqrt{C + \alpha\alpha + \frac{AE}{\alpha\alpha}} \tag{7}$$

quae simul est integralis completa:

§. 19. Vel ponamus $A = f\alpha\alpha$: $C = g\alpha\alpha$ et $E = b\alpha\alpha$, vt habeamus hanc aequationem differentialem

$$\frac{dx}{\sqrt{f + gxx + bx^4}} = \frac{dy}{\sqrt{f + gyy + by^4}}$$

cuius propterea aequatio integralis completa erit:

$$xx + yy = f + bxxyy + 2xy\sqrt{1 + g + fb}$$

quae etsi nouam constantem inuoluere non videtur, tamen est completa, cum in differentiali tantum ratio quantitatum f, g, et b spectatur, ita vt pro f, g, et b scribere liceat fcc, gcc et bcc, vnde aequatio integralis manifesto completa prodit:

$$xx + yy = fcc + bccxxyy + 2xy\sqrt{1 + gcc + fbc^4}$$

vel $f(xx + yy) = fee + beexxyy + 2xy\sqrt{f(f + gee + be^4)}$ posito $cc = \dfrac{ee}{f}$.

§. 20. Quodsi ergo proposita sit haec aequatio differentialis

$$\frac{dx}{\sqrt{f + gxx + bx^4}} = \frac{dy}{\sqrt{f + gyy + by^4}}$$

valor ipsius y per functionem algebraicam ipsius x exprimi poterit, ita vt sit:

$$y = \frac{x\sqrt{1 + gcc + fbc^4} \pm c\sqrt{f + gxx + bx^4}}{1 - bccxx}$$

vel $y = \dfrac{x\sqrt{f(f + gee + be^4)} \pm e\sqrt{f(f + gxx + bx^4)}}{f - beexx}$.

Quodsi ergo sit $g = 0$, vt habeatur haec aequatio differentialis

$$\frac{dx}{\sqrt{f + bx^4}} = \frac{dy}{\sqrt{f + by^4}}$$

valor integralis completus ipsius y erit

$$y = \frac{x\sqrt{f(f + be^4)} \pm e\sqrt{f(f + bx^4)}}{f - beexx}$$

vnde constantem e pro lubitu determinando, innumeri valores particulares pro y deduci possunt.

§. 21. Methodi autem, qua supra vsus sum, beneficio etiam huius aequationis

$$\frac{m\,dx}{\sqrt{f + gxx + bx^4}} = \frac{n\,dy}{\sqrt{f + gyy + by^4}}$$

si modo m et n sint numeri rationales, integrale completum, atque id quidem algebraice, exhiberi poterit.

§. 22. Quemadmodum in aequatione supra assumta, variabiles x et y inter se permutabiles sunt constitutae, vt ambae formulae inter se similes euaderent, ita omissa hac limitatione ad formularum differentialium disparium comparationem perueniemus. Ponamus ergo:

$$\alpha xx + \beta yy = 2\gamma xy + \delta xxyy + \varepsilon \tag{1}$$

vnde fit

$$x = \frac{\gamma y + \sqrt{\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4}}{\alpha - \delta yy}$$

et $y = \dfrac{\gamma x - \sqrt{\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4}}{\beta - \delta xx}$ hincque

$$\alpha x - \gamma y - \delta xyy = \sqrt{\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4} \tag{2}$$

$$\beta y - \gamma x - \delta xxy = -\sqrt{\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4} \tag{3}$$

at aequatio (1) differentiata dat:

$$dx(\alpha x - \gamma y - \delta xyy) + dy(\beta y - \gamma x - \delta xxy) = 0$$

vnde conficitur haec aequatio differentialis:

$$\frac{dx}{\sqrt{\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4}} = \frac{dy}{\sqrt{\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4}}$$

cuius propterea integralis est aequatio assumta.

§. 23. Verum haec disparitas facile tollitur, loco y ponendo $z\sqrt{\dfrac{\alpha}{\beta}}$, cuius rei ratio statim ex aequatione assumta potuisset esse manifesta. Sed alia patet via ad formulas dispares perueniendi, cuius hic exemplum tradidisse sufficiat. Assumatur aequatio: $x^4 + 2axxyy + 2bxx \ = c$, cuius differentiale est $dx(x^3 + axyy + bx) + axxy\,dy = 0$, seu

$$\frac{dx}{xy} = \frac{-a\,dy}{xx + ayy + b}$$

Iam ex aequatione assumta primo determinetur xy per x sicque fiet $xy = \sqrt{\dfrac{c - 2bxx - x^4}{2a}}$; tum vero $xx + ayy + b$ per y, at ob $(xx + ayy + b)^2 = c + (ayy + b)^2$, erit

$$xx + ayy + b = \sqrt{c + (ayy + b)^2}$$

Quocirca habebitur aequatio differentialis ista

$$\frac{dx\sqrt{2a}}{\sqrt{c - 2bxx - x^4}} = \frac{a\,dy}{\sqrt{c + bb + 2abyy + aay^4}}$$

cuius propterea integralis est assumta seu $y = \dfrac{\sqrt{c - 2bxx - x^4}}{x\sqrt{2a}}$.

§. 24. Etsi hoc integrale non est completum, tamen ex superioribus facile completum reddetur. Ponatur enim:

$$\frac{a\,dy}{\sqrt{c + bb + 2abyy + aay^4}} = \frac{a\,dz}{\sqrt{c + bb + 2abzz + aaz^4}}$$

ob $f = c + bb$; $g = 2ab$; $b = aa$, erit

$$y = \frac{z\sqrt{(c + bb)(c + bb + 2abee + aae^4)} \pm e\sqrt{(c + bb)(c + bb + 2abzz + aaz^4)}}{c + bb - aaeezz}$$

hic ergo valor aequalis statuatur ipsi $\dfrac{\sqrt{c - 2bxx - x^4}}{x\sqrt{2a}}$, et aequatio hinc inter x et z resultans integralis erit completa huius aequationi differentialis

$$\frac{dx\sqrt{2a}}{\sqrt{c - 2bxx - x^4}} = \frac{a\,dz}{\sqrt{c + bb + 2abzz + aaz^4}}$$

Quin etiam ex allatis patet, si haec bina membra insuper per numeros rationales quoscunque multiplicentur, quemadmodum integrale completum inueniri oporteat.

§. 25. Verum missa membrorum disparitate formationem parium membrorum generalius concipiamus, ponatur ergo:

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \zeta xxyy \qquad (1)$$

vnde differentiando obtinetur:

$$dx(\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy) + dy(\beta + \gamma y + \delta x + 2\varepsilon xy + \varepsilon xx + \zeta xxy) = 0$$

ideoque

$$\frac{dy}{\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy} = \frac{-dx}{\beta + \gamma y + \delta x + 2\varepsilon xy + \varepsilon xx + \zeta xxy} \qquad (2)$$

Ex resolutione autem aequationis assumtae elicitur.

$$y = \frac{-\beta - \delta x - \varepsilon xx + \sqrt{\beta\beta - \alpha\gamma + 2(\beta\delta - \alpha\varepsilon - \beta\gamma)x + (\delta\delta - \gamma\gamma - \alpha\zeta - 2\beta\varepsilon)xx + 2(\delta\varepsilon - \beta\zeta - \gamma\varepsilon)x^3 + (\varepsilon\varepsilon - \gamma}}{\gamma + 2\varepsilon x + \zeta xx}$$

Ponatur breuitatis gratia

$$\beta\beta - \alpha\gamma = A; \quad \beta\delta - \alpha\varepsilon - \beta\gamma = B; \quad \delta\delta - \gamma\gamma - \alpha\zeta - 2\beta\varepsilon = C; \quad \varepsilon\varepsilon - \gamma\zeta = E; \quad \delta\varepsilon - \beta\zeta - \gamma\varepsilon = D;$$

eritque

$$\beta + \delta x + \varepsilon xx + \gamma y + 2\varepsilon xy + \zeta xxy = \pm\sqrt{A + 2Bx + Cxx + 2Dx^3 + Ex^4}$$

$$\beta+\delta y+\varepsilon yy+\gamma x+2\varepsilon xy+\zeta xyy=\mp\sqrt{A+2By+Cyy+2Dy^3+Ey^4}$$

§. 26. Hinc itaque concludimus huius aequationis differentialis:

$$\frac{dx}{\sqrt{A+2Bx+Cxx+2Dx^3+Ex^4}}=\frac{dy}{\sqrt{A+2By+Cyy+2Dy^3+Ey^4}}$$

aequationem integralem eamque completam esse

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy$$

adhibita scilicet superiori horum coëfficientium determinatione. Primum autem definiatur β vel ε ex hac aequatione

$$\frac{BB(\varepsilon\varepsilon-E)-DD(\beta\beta-A)}{A\varepsilon\varepsilon-E\beta\beta}+\frac{2AD\varepsilon-2BE\beta}{B\varepsilon-D\beta}=C$$

tum vero erit:

$$\gamma=\frac{A\varepsilon\varepsilon-E\beta\beta}{B\varepsilon-D\beta};\quad \alpha=\frac{\beta\beta-A}{\gamma};\quad \zeta=\frac{\varepsilon\varepsilon-E}{\gamma}\quad \text{et}$$

$$\delta=\frac{B\beta(\varepsilon\varepsilon-E)-D\varepsilon(\beta\beta-A)}{A\varepsilon\varepsilon-E\beta\beta}+\gamma\quad \text{seu}\quad \delta=\gamma+\frac{B+\alpha\varepsilon}{\beta}$$

§. 27. Hinc ergo perspicuum est etiam hanc aequationem differentialem:

$$\frac{dx}{\sqrt{A+2Dx^3}}=\frac{dy}{\sqrt{A+2Dy^3}}$$

integrari posse: nam ob $B=0$, $C=0$ et $E=0$ erit

$$-\frac{DD(\beta\beta-A)}{A\varepsilon\varepsilon}-\frac{2A\varepsilon}{\beta}=0\quad \text{seu}\quad \varepsilon=\sqrt[3]{\frac{DD}{2AA}\beta(A-\beta\beta)}$$

at hinc valores nimis prodeunt complicati. Facilius negotium absoluetur, resoluendo valores litterarum euanescentium B, C et E; nam $E=0$ dat: $\zeta=\frac{\varepsilon\varepsilon}{\gamma}$; tum $B=0$ dat: $\delta=\gamma+\frac{\alpha\varepsilon}{\beta}$; atque $C=0$ dat $\delta\delta-\gamma\gamma=\alpha\zeta+2\beta\varepsilon=\frac{\alpha\varepsilon\varepsilon}{\gamma}+2\beta\varepsilon=\frac{\alpha^2\varepsilon\varepsilon}{\beta\beta}+\frac{2\alpha\gamma\varepsilon}{\beta}$ cuius factores sunt $\beta\beta=\alpha\gamma$ et $\alpha\varepsilon\varepsilon+2\beta\gamma\varepsilon=0$. At si esset $\beta\beta=\alpha\gamma$ foret $A=0$, sin autem esset $\varepsilon=0$ foret et $\zeta=0$ et $D=0$, contra scopum. Fieri ergo oportet $\alpha\varepsilon=-2\beta\gamma$; vnde fiet $\alpha=-\frac{2\beta\gamma}{\varepsilon}$; $\delta=-\gamma$; et $\zeta=\frac{\varepsilon\varepsilon}{\gamma}$. Denique fieri debet $\beta\beta+\frac{2\beta\gamma\gamma}{\varepsilon}=A$ et $-2\gamma\varepsilon-\frac{\beta\varepsilon\varepsilon}{\gamma}=D$. Inde fit $\varepsilon=\frac{2\beta\gamma\gamma}{A-\beta\beta}$; et ob $\frac{\gamma D}{\varepsilon}=-(2\gamma\gamma+\beta\varepsilon)$ et $2\gamma\gamma+\beta\varepsilon=\frac{A\varepsilon}{\beta}$, erit $\frac{\gamma D}{\varepsilon}=-\frac{A\varepsilon}{\beta}$; ideoque $\varepsilon\varepsilon=-\frac{\beta\gamma D}{A}$. Ergo

$$\frac{4\beta\gamma^3}{(A-\beta\beta)^2}+\frac{D}{A}=0.$$

§. 28. Cum autem tantum ratio litterarum A et D in censum veniat, aequatio vltima valori absoluto ipsius A inueniendo inseruit, quem autem nosse non est opus. Manebunt ergo litterae γ et β indeterminatae. Ponatur ergo $\gamma = -Ac$ et $\beta = Dc$, erit $\varepsilon\varepsilon = DDcc$, seu $\varepsilon = Dc$, hincque $\delta = Ac$; $\zeta = -\frac{DDc}{A}$; et $\alpha = 2Ac$. Quare huius aequationis differentialis:

$$\frac{dx}{\sqrt{A+2Dx^3}} = \frac{dy}{\sqrt{A+2Dy^3}}$$

integrale est.

$$0 = 2A + 2D(x+y) - A(xx+yy) + 2Axy + 2Dxy(x+y) - \frac{DD}{A}xxyy$$

Hoc autem integrale non est completum, tale autem reddetur ponendo $\gamma = -A$ et $\beta = Dcc$, vnde fit $\varepsilon\varepsilon = DDcc$ et $\varepsilon = Dc$; porro erit $\delta = A$; $\zeta = -\frac{DDcc}{A}$; $\alpha = 2Ac$; ita vt integrale completum sit:

$$0 = 2Ac + 2Dcc(x+y) - A(xx+yy) + 2Axy + 2Dcxy(x+y) - \frac{DDcc}{A}xxyy$$

vbi c est constans ab arbitrio pendens, vnde fit:

$$y = \frac{Dcc + Ax + Dcxx + \sqrt{c\big(2A + \frac{DD}{A}c^3\big)(A+2Dx^3)}}{A - 2Dcx + \frac{DDcc}{A}xx}$$

§. 29. Hic casus notari meretur, quo $A = 1$ et $D = \frac{1}{2}$, vt habeatur haec aequatio differentialis

$$\frac{dx}{\sqrt{1+x^3}} = \frac{dy}{\sqrt{1+y^3}}$$

vbi ad fractiones tollendas loco c scribatur $2c$ eritque integrale completum:

$$0 = 4c + 4cc(x+y) - xx - yy + 2xy + 2cxy(x+y) - ccxxyy$$

seu $y = \dfrac{2cc + x + cxx + 2\sqrt{c(1+c^3)(1+x^3)}}{1 - 2cx + ccxx}$

Integralia ergo particularia erunt

I. si $c = 0$; $y = x$;

II. si $c = \infty$; $y = \dfrac{2 \pm 2\sqrt{1+x^3}}{xx}$;

III. si $c = -1$; $y = \dfrac{2 + x - xx}{1 + 2x + xx} = \dfrac{2-x}{1+x}$.

§. 30. Ex eodem principio si in §. 29. loco litterarum A, B, C, D, E, eaedem per quantitatem quampiam p multiplicentur, nihilo minus aequatio differentialis erit

$$\frac{dx}{\sqrt{A+2Bx+Cxx+2Dx^3+Ex^4}} = \frac{dy}{\sqrt{A+2By+Cyy+2Dy^3+Ey^4}}$$

inuenieturque

$$p = \frac{BB\varepsilon\varepsilon - DD\beta\beta}{BBE - ADD} + 2\frac{(AD\varepsilon - BE\beta)(A\varepsilon\varepsilon - E\beta\beta)}{(B\varepsilon - D\beta)(BBE - ADD)} - \frac{C(A\varepsilon\varepsilon - E\beta\beta)}{BBE - ADD}$$

tum erit $\gamma = \dfrac{A\varepsilon\varepsilon - E\beta\beta}{B\varepsilon - D\beta}$; $\alpha = \dfrac{\beta\beta - Ap}{\gamma}$; $\zeta = \dfrac{\varepsilon\varepsilon - Ep}{\gamma}$ atque $\delta = \gamma + \dfrac{\alpha\varepsilon + Bp}{\beta}$: ita vt litterae β et ε maneant indeterminatae, fietque propterea aequatio integralis completa:

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy$$

vnde fit:

$$y = \frac{-\beta - \delta x - \varepsilon xx + \sqrt{p(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}{\gamma + 2\varepsilon x + \zeta xx}$$

§. 31. Notandum denique est, non solum hanc aequationem differentialem, cuius integrale completum modo exhibui, sed etiam hanc multo latius patentem

$$\frac{m\,dx}{\sqrt{A + 2Bx + Cxx + 2Dx^3 + Ex^4}} = \frac{n\,dy}{\sqrt{A + 2By + Cyy + 2Dy^3 + Ey^4}}$$

semper algebraice et quidem complete integrari posse, dummodo coëfficientium m et n ratio fuerit rationalis: haec enim integratio simili modo instituitur, quo supra vsus sum ad aequationem, quae mihi hic praecipue erat proposita, integrandam. Methodus autem, cuius hic specimina attuli, ita mihi videtur comparata, vt indolem eius diligentius excolendo, ad insignes vsus apta reddi queat, vnde haud contemnenda commoda in Analysin sint redundatura.

§. 32. Hic autem obseruo, formulam §. 28 assumtam latius extendendo, eiusmodi differentialia inter se comparari posse, quae sint disparia, atque adeo exemplum disparitatis §. 26. allatum hoc modo obtineri posse; ita vt omnia, quae hactenus sunt tradita, in hac generali inuestigatione contineantur. Fingatur scilicet haec aequatio integralis:

$$\alpha xxyy + 2\beta xxy + 2\gamma xyy + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \varkappa = 0 \tag{1}$$

ex qua fit

$$y = \frac{-\beta xx - \zeta x - \theta + \sqrt{(\beta xx + \zeta x + \theta)^2 - (\alpha xx + 2\gamma x + \varepsilon)(\delta xx + 2\eta x + \varkappa)}}{\alpha xx + 2\gamma x + \varepsilon} \tag{2}$$

$$x = \frac{-\gamma yy - \zeta y - \eta - \sqrt{(\gamma yy + \zeta y + \eta)^2 - (\alpha yy + 2\beta y + \delta)(\varepsilon yy + 2\theta y + \varkappa)}}{\alpha yy + 2\beta y + \delta} \tag{3}$$

Ponatur iam breuitatis gratia:

$$\begin{aligned} App &= \beta\beta - \alpha\delta & \mathfrak{A}qq &= \gamma\gamma - \alpha\varepsilon \\ 2Bpp &= 2\beta\zeta - 2\alpha\eta - 2\gamma\delta & 2\mathfrak{B}qq &= 2\gamma\zeta - 2\alpha\theta - 2\beta\varepsilon \\ Cpp &= \zeta\zeta + 2\beta\theta - \alpha\varkappa - \delta\varepsilon - 4\gamma\eta & \mathfrak{C}qq &= \zeta\zeta + 2\gamma\eta - \alpha\varkappa - \delta\varepsilon - 4\beta\theta \\ 2Dpp &= 2\zeta\theta - 2\gamma\varkappa - 2\varepsilon\eta & 2\mathfrak{D}qq &= 2\zeta\eta - 2\beta\varkappa - 2\delta\theta \\ Epp &= \theta\theta - \varepsilon\varkappa & \mathfrak{E}qq &= \eta\eta - \delta\varkappa \end{aligned}$$

eritque:

$$p\sqrt{Ax^4+2Bx^3+Cxx+2Dx+E}=\alpha xxy+2\gamma xy+\varepsilon y+\beta xx+\zeta x+\theta \quad (4)$$

$$-q\sqrt{\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}}=\alpha xyy+2\beta xy+\delta x+\gamma yy+\zeta y+\eta \quad (5)$$

§. 33. At si aequatio integralis assumta differentietur, fiet

$$dx(\alpha xyy+2\beta xy+\gamma yy+\delta x+\zeta y+\eta)+dy(\alpha xxy+\beta xx+2\gamma xy+\varepsilon y+\zeta x+\theta)=0 \quad (6)$$

vnde si istorum factorum valores (4) et (5) reperti substituantur, orietur ista aequatio differentialis:

$$\frac{q\,dx}{\sqrt{Ax^4+2Bx^3+Cxx+2Dx+E}}=\frac{p\,dy}{\sqrt{\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}}} \quad (7)$$

cuius propterea integralis est aequatio assumta (1). Cum autem supra habeantur 10 aequationes, coëfficientium autem α, β, γ, δ, etc. numerus sit 9, quorum vnus pro lubitu assumi potest, octo remanebunt litterae determinandae. Porro autem insuper definiendae accedunt binae litterae p et q, ita vt nunc decem quantitates adsint incognitae; ex quo coëfficientes vtriusque formulae A, B, C, D, E et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ videntur pro lubitu assumi posse. Verum perspicuum est, cum alteri iam fuerint ad libitum assumti, alteros non omnino ab arbitrio nostro pendere, alias enim quaeuis formula ad algebraicam reduci posset.

§. 34. Hinc autem aliae datae formulae transmutationes non inelegantes obtineri possunt, si loco y alii valores substituantur. Veluti si ponatur $\mathfrak{C}=0$,[2] seu $\eta\eta=\delta\varkappa$, statuaturque $y=zz$ sequens prodibit aequatio differentialis.

$$\frac{q\,dx}{\sqrt{Ax^4+2Bx^3+Cxx+2Dx+E}}=\frac{2p\,dz}{\sqrt{\mathfrak{A}z^6+2\mathfrak{B}z^4+\mathfrak{C}z^2+2\mathfrak{D}}} \quad (8)$$

cuius propterea integralis est aequatio assumta, si ponatur $y=zz$, statuaturque $\eta\eta=\delta\varkappa$, ac reliquae litterae rite determinentur. Integrale etiam completum nulla difficultate reperietur, nam etiamsi fortasse integrale inuentum nouam non inuoluat constantem, ponatur

$$\frac{q\,dx}{\sqrt{Ax^4+2Bx^3+Cxx+2Dx+E}}=\frac{q\,du}{\sqrt{Au^4+2Bu^3+Cuu+2Du+E}}$$

et huius aequationis integrale completum ex antecedentibus assignare licebit; atque hinc integrale quoque completum aequationis ex formulis disparibus constantis colligetur.

§. 35. Quemadmodum huius aequationis differentialis, vt a simplicissimis incipiam:

$$\frac{dx}{\sqrt{f+gx}}=\frac{dy}{\sqrt{f+gy}}$$

[2] *Editorial note:* The text sets $\mathfrak{C}=0$, but the gloss $\eta\eta=\delta\varkappa$ is in fact the condition $\mathfrak{E}=0$ (since $\mathfrak{E}qq=\eta\eta-\delta\varkappa$); equation (8) that follows holds only when $\mathfrak{E}=0$. An apparent misprint for $\mathfrak{E}=0$, reproduced here as printed.

integrale completum est:

$$gg(xx+yy)-2ggxy-2ccg(x+y)+c^4-4ccf=0$$

Deinde vero huius aequationis differentialis

$$\frac{dx}{\sqrt{f+gxx}}=\frac{dy}{\sqrt{f+gyy}}$$

integrale completum est:

$$xx+yy-2xy\sqrt{1+fgcc}-ccff=0$$

Tertio vero huius aequationis differentialis

$$\frac{dx}{\sqrt{f+gx^3}}=\frac{dy}{\sqrt{f+gy^3}}$$

integrale completum est

$$f(xx+yy)+\frac{ggcc}{4f}xxyy-gcxy(x+y)-2fxy-gcc(x+y)-2fc=0$$

Quarto porro huius aequationis differentialis

$$\frac{dx}{\sqrt{f+gx^4}}=\frac{dy}{\sqrt{f+gy^4}}$$

integrale completum repertum est

$$f(xx+yy)-fcc-gccxxyy-2xy\sqrt{f(f+gc^4)}=0$$

Ita etiam integrale completum huius aequationis

$$\frac{dx}{\sqrt{f+gx^6}}=\frac{dy}{\sqrt{f+gy^6}}$$

reperiri poterit:

§. 36. Determinentur primo in §. 33. valores, ita vt prodeat haec aequatio

$$\frac{dx}{\sqrt{fx+gx^4}}=\frac{dy}{\sqrt{fy+gy^4}}$$

cuius integralis completa reperitur:

$$gg(xx+yy)-4ggcxxyy-4fgccxy(x+y)-2ggxy-2fgc(x+y)+ffcc=0$$

Ponatur nunc $x=tt$ et $y=uu$, vt prodeat haec aequatio differentialis

$$\frac{dt}{\sqrt{f+gt^6}}=\frac{du}{\sqrt{f+gu^6}}$$

cuius propterea integralis completa erit

$$gg(t^4+u^4)-4ggct^4u^4-4fgccttuu(tt+uu)-2ggttuu-2fgc(tt+uu)+ffcc=0$$

vnde notari meretur casus ex hypothesi $c=\infty$ resultans, qui dat $4gttuu(tt+uu)=f$.