XXXIV. Method for measuring the lemniscate

Sketch II (*)

(*) Giornale de' letterati d'Italia, vol. XXX, p. 87.

Theorem I. — Let the two equations (1) and (2) written below be given; I say that, the first of them being posited, the other also holds

$$x = \frac{\sqrt{1 \mp \sqrt{1-z^4}}}{z}. \tag{1}$$

$$\frac{\pm dz}{\sqrt{1-z^4}} = \frac{dx\sqrt{2}}{\sqrt{1+x^4}}. \tag{2}$$

Proof. — Equation (1), differentiated, furnishes a value of dx which, being reduced to a simple expression, gives

$$dx = \frac{\pm dz\sqrt{1 \mp \sqrt{1-z^4}}}{z^2\sqrt{1-z^4}}.$$

From the same equation (1) one further deduces the following

$$\frac{\sqrt{1+x^4}}{\sqrt{2}} = \frac{\sqrt{1 \mp \sqrt{1-z^4}}}{z^2}.$$

And dividing the second-to-last equation by this last one, one arrives at equation (2).

Which was to be demonstrated.

Theorem II. — Let the two equations (3) and (4) written below be given; I say that, the first of it being posited, the other also holds

$$x = \frac{\sqrt{1 \mp z}}{\sqrt{1 \pm z}}. \tag{3}$$

$$\frac{\mp dz}{\sqrt{1-z^4}} = \frac{dx\sqrt{2}}{\sqrt{1+x^4}}. \tag{4}$$

Proof. — Differentiating equation (3), and proceeding in the proper manner, one has $dx = \dfrac{\pm dz}{\sqrt{1-z^2}} \times \dfrac{1}{1 \pm z}$.

Moreover the same equation (3) shows $\dfrac{\sqrt{1+x^4}}{\sqrt{2}} = \dfrac{\sqrt{1+z^2}}{1 \pm z}$.

And dividing the second-to-last equation by this last one, one obtains equation (4).

Which was to be demonstrated.

Theorem III. — Let the two equations (5) and (6) written below be given; I say that, the first of them being posited, the other also holds

$$x = \frac{u\sqrt{2}}{\sqrt{1-u^4}}. \tag{5}$$

$$\frac{du}{\sqrt{1-u^4}} = \frac{1}{\sqrt{2}} \times \frac{dx}{\sqrt{1+x^4}}. \tag{6}$$

Proof. — From equation (5), differentiated and duly handled, there results $\frac{dx}{\sqrt{2}} = \frac{du}{\sqrt{1-u^4}} \times \frac{1+u^4}{1-u^4}$, and from the said equation (5) one deduces $\sqrt{1+x^4} = \frac{1+u^4}{1-u^4}$.

Dividing thereafter the second-to-last equation by this last one, one arrives at equation (6).

Which was to be demonstrated.

Theorem IV. — Let the two equations (6) and (8) written below be given; I say that, the first of them being posited, the other also holds

$$x = \frac{\sqrt{1-t^4}}{t\sqrt{2}}. \tag{7}$$

$$\frac{-dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}} \times \frac{dx}{\sqrt{1+x^4}}. \tag{8}$$

Proof. — Differentiating equation (7), and reducing the value of dx to a convenient expression, one finds $dx\sqrt{2} = \frac{-dt}{\sqrt{1-t^4}} \times \frac{1+t^4}{t^2}$.

The same equation (7) shows $2\sqrt{1+x^4} = \frac{1+t^4}{t^2}$.

And the second-to-last equation, divided by this last one, yields equation (8).

Which was to be demonstrated.

Scholium I. — To make use of these theorems let it be noted:

First, that $\int \frac{dz}{\sqrt{1-z^4}}$ represents the direct arc CS of the lemniscate (fig. 29) subtended by the chord $CS = z$, provided the semi-axis CL of this curve is equal to unity.

Second, that $\int \frac{-dz}{\sqrt{1-z^4}}$ expresses the inverse arc SL of the same curve, which corresponds to the aforesaid chord CS (z).

Third, that $\int \frac{dz}{\sqrt{1+x^4}} = \frac{3}{2}\int dx\,\sqrt{1+x^4} - \frac{1}{2}\sqrt{1+x^4}$, as is proved by calculation.

Fourth, that $\int dx\,\sqrt{1+x^4}$ expresses the arc AQ (fig. 30) of the primary cubic parabola corresponding to the abscissa $AF = x$, provided that, taking a as unity, the parameter of this parabola is $a\sqrt{3}$, and the ordinates are normal to the abscissa.

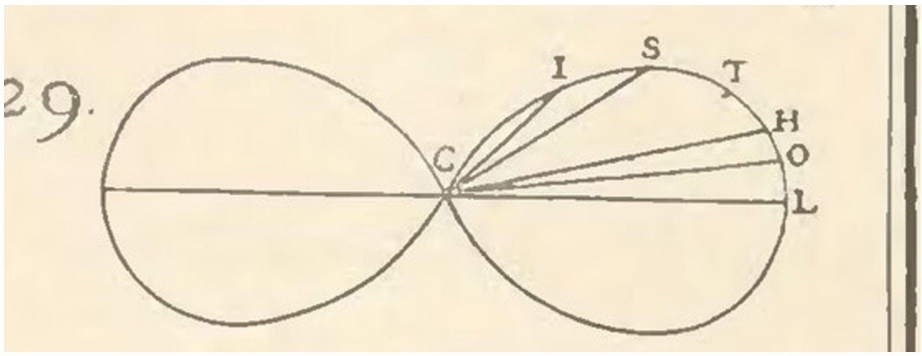

Figure 1: Fig. 29.

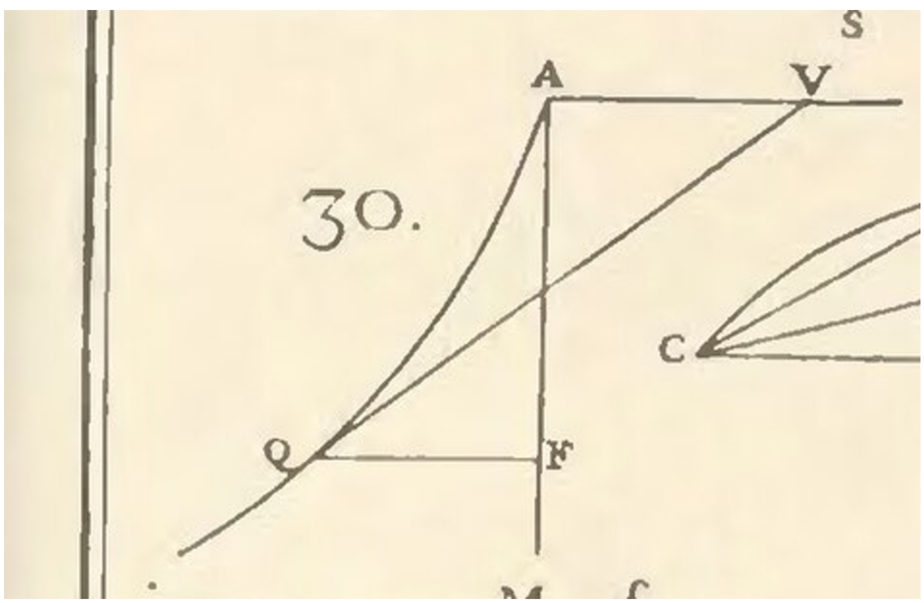

Figure 2: Fig. 30.

Fifth, that $x\sqrt{1+x^4}$ denotes the portion QV of the tangent (fig. 30) contained between the ordinate FQ and the indefinite line AV, which from the origin A of the parabola runs parallel to the ordinates.

Sixth, that in consequence one will have

$$\int \frac{dx}{\sqrt{1+x^4}} = \frac{3}{2}\,\text{arc.}\,AQ - \frac{1}{2}\,QV.$$

Examples to show the way of applying the preceding theorems to geometry.

Example I, for Theorem I (fig. 29 and 30). — In equation (1) let the arbitrary sign $\mp$ signify the negative sign; let equation (2) be integrated by means of the preceding scholium, and one will find

$$\text{Arc.}\,CS = \frac{3}{\sqrt{2}}\,\text{arc.}\,AQ - \frac{1}{\sqrt{2}}\,QV.$$

Annulling $AF\,(x)$ will also annul $CS\,(z)$, and so this equation is complete.

Example II, for Theorem II (fig. 29 and 30). — In equation (3) let the sign $\mp$ represent the negative sign; let equation (4) be integrated, and one will discover

$$\text{Arc.}\,SL = \frac{3}{\sqrt{2}}\,\text{arc.}\,AQ - \frac{1}{\sqrt{2}}\,QV.$$

The supposition $CS\,(z) = 1$, which annihilates the inverse arc SL, renders $AF\,(x) = 0$, and so this equation is complete.

Scholium II. — In the course of this paper I shall no longer prove that the equations are complete, it being enough for readers to have the method by which I have proceeded in these two examples, so that they may satisfy themselves of the completeness of the following equations; they will also be able to make use of the four preceding theorems to draw out other ways of measuring the lemniscate by means of the extent of the primary cubic parabola, not omitting, where it is needed, the subtraction or addition of the constant quantity proper to render the equations complete.

They will moreover be able to deduce from these theorems truths altogether new, concerning the comparison of the arcs of the said parabola, etc.

I shall content myself with observing that, since the measure of the primary cubic parabola depends in several ways on the extent of the lemniscate by virtue of the four preceding theorems, and the measure of the lemniscate depends on the extent of the equilateral hyperbola and of a kind of ellipse *jointly*, as I discovered in Sketch I, it follows that the measure of the primary cubic parabola depends in several ways on the extent of the two aforesaid conic sections *jointly*. This invention cannot fail to please whoever shall have considered what may be read in the Acta of Leipzig of the year 1695, p. 64, and p. 184 toward the end.

Theorem V. — Let the two equations (9) and (10) written below be given; I say that, the first of them being posited, the other also holds

$$\frac{u\sqrt{2}}{\sqrt{1-u^4}} = \frac{1}{z}\sqrt{1-\sqrt{1-z^4}}. \tag{9}$$

$$\frac{dz}{\sqrt{1-z^4}} = \frac{2\,du}{\sqrt{1-u^4}}. \tag{10}$$

Proof. — Setting in equation (1) of Theorem I the negative sign in place of $\mp$, and in place of x its value $\dfrac{u\sqrt{2}}{\sqrt{1-u^4}}$ noted in equation (5) of Theorem III, one has equation (9); then sur- rogating in equation (2) of Theorem I, in place of $\dfrac{dx\sqrt{2}}{\sqrt{1+x^4}}$, its value $\dfrac{2\,du}{\sqrt{1-u^4}}$, which arises from the supposition made of x (as is seen by multiplying equation (6) of Theorem III by 2), equation (10) follows.

Which was to be demonstrated.

Corollary I (fig. 29). — Let z be called the chord CS of the lemniscate, and u the other chord CI, and integrating equation (10), one will find

$$\text{Arc.}\, CS = 2\,\text{arc.}\, CI.$$

Whence, considering the letter z as known in equation (9), one will draw from it the value of the chord $CI\,(u)$ that bisects the direct arc CS.

Corollary II (fig. 29). — From equation (9) arises this other

$$z = \frac{2u\sqrt{1-u^4}}{1+u^4}. \tag{11}$$

And this value of z is such that, calling u the chord CI taken as known, and making the other chord CS equal to the second member of equation (11), the direct arc CS is double the other direct arc CI.

Theorem IV. — Let the two equations (12) and (13) written below be given; I say that, the first of them being posited, the other also holds

$$\frac{\sqrt{1-t^4}}{t\sqrt{2}} = \frac{1}{z}\sqrt{1-\sqrt{1-z^4}}. \tag{12}$$

$$\frac{dz}{\sqrt{1-z^4}} = -\frac{2\,dt}{\sqrt{1-t^4}}. \tag{13}$$

Proof. — Setting in equation (1) of Theorem I the negative sign in place of $\mp$, and in place of x its value $\dfrac{1}{t\sqrt{2}}\sqrt{1-t^4}$ noted in equation (7) of Theorem IV, one has equation (12). Then substi- tuting in equation (2) of Theorem I, in place of $\dfrac{dx\sqrt{2}}{\sqrt{1+x^4}}$, its value $\dfrac{-2\,dt}{\sqrt{1-t^4}}$, which derives from the supposition made of x (as appears by multiplying equation (8) of Theorem IV by 2), one obtains equation (13).

Which was to be demonstrated.

Corollary I (fig. 29). — Let z be named the chord CS of the lemniscate, and (t) the other chord CH, and there will result arc. $CS = 2$ arc. HL.

Corollary II (fig. 29). — Rearranging equation (12), and putting in it u in place of z, and r in place of t, one arrives at this other

$$u = \frac{2r\sqrt{1-r^4}}{1+r^4}. \tag{14}$$

So that, naming r the chord CO, and taking the other chord $CI\,(u)$ equal to the second member of equation (14), the direct arc CI will be double the inverse arc OL.

Problem I. — To cut the quadrant of the lemniscate into three equal parts.

Solution (fig. 29 and 31). — By virtue of Theorem VI and its Corollary I, the direct arc CS (fig. 29) will be double the inverse arc HL, provided that, calling t the chord CH, and z the other chord CS, one has equation (12). Suppose now that the two points S and H coincide in T; this supposition will render $CS\,(z) = CH\,(t)$, and the direct arc CT (fig. 31) will be double the inverse arc TL, which will become the third part of the quadrant.

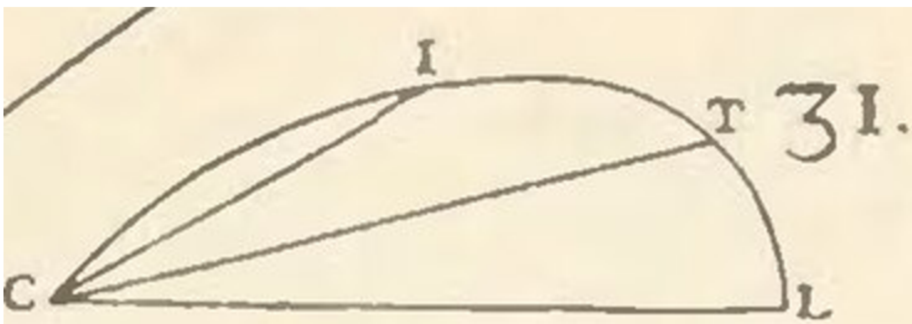

Figure 3: Fig. 31.

But supposing $t = z$ in equation (12), and thereafter arranging it in the proper manner, etc., one obtains

$$z = \sqrt[4]{-3+2\sqrt{3}}.$$

Whence, attributing to the chord CT (fig. 31) this value of z, and taking the direct arc CI equal to the inverse arc TL by means of Theorem III of Sketch I, one will have the solution of the problem.

Which was to be found.

Problem II. — To cut the quadrant of the lemniscate into five equal parts.

Solution (fig. 29 and 31). — Let u be called the chord CI (fig. 29) of the lemniscate, and z the other chord CS; let $z = \frac{2u\sqrt{1-u^4}}{1+u^4}$, and the arc CS will be double the arc CI by the second corollary of Theorem V. Let r then be named the chord CO (fig. 29); suppose the chord $CI\,(u) = \frac{2r\sqrt{1-r^4}}{1+r^4}$, and the direct arc CI will be double the inverse arc OL by Corollary II of Theorem VI; therefore the direct arc CS will be quadruple the inverse arc OL.

Suppose now $CS\,(z) = CO\,(r)$; the points S and O will coincide in T, the arc TL (fig. 32) will be one fifth of the quadrant, the arc $CI = 2$ arc. TL will contain two fifths of the same quadrant, and the arc $IT =$ arc. CI will contain the other two fifths of it.

Let there then be taken, by means of Theorem III of Sketch I, the inverse arc EL (fig. 32) equal to the direct arc CI, and the direct arc CB equal to the inverse arc TL, and the points B, I, E, T will cut the quadrant into five equal parts.

It remains only to find the precise value of the chord CT (fig. 32), equal at the same time to z and to r, and this will be found by substituting in equation (14), in place of r, the letter z, whence one will have

$$u = \frac{2z\sqrt{1-z^4}}{1+z^4}. \tag{15}$$

And putting this value of u in equation (11), one will arrive at another equation which will contain no unknown but the letter z, and which must be reckoned an equation of the eighth degree, since if in it one supposes $z^4 = b$, there will result a new equation of exactly the eighth degree; the calculation will be longer than difficult, provided one does not neglect those ways of facilitating it which are well known to skilled analysts, and so I judge it superfluous to set it out in this paper. I shall only remark that, in the equation expressing the value of z^4, the same quantity z^4 will have two true values less than a^4; the greater of these two values will express the fourth power of $CT\,(z)$, and the lesser of them will express the fourth power of $CI\,(u)$; for if in equation (15) one puts, in place of z, its value in u drawn from equation (11), one will find an equation altogether similar to that which expresses the value of z^4, etc.

Which was to be found.

The late Monsieur the Marquis de l'Hospital, in his excellent *Treatise on Conic Sections*, book 9, prop. 9, teaches a way of constructing equations of the eighth degree.

Theorem VII. — Let the two equations (16) and (17) written below be given; I say that, the first of them being posited, the other also holds

$$\frac{u\sqrt{2}}{\sqrt{1-u^4}} = \frac{\sqrt{1-z}}{\sqrt{1+z}}. \tag{16}$$

$$\frac{-dz}{\sqrt{1-z^4}} = \frac{2\,du}{\sqrt{1-u^4}}. \tag{17}$$

Which was to be demonstrated.

Proof. — In equation (3) of Theorem II let the sign $\mp$ represent the negative sign, and in place of x let there be put its value $\dfrac{u\sqrt{2}}{\sqrt{1-u^4}}$ noted in equation (5) of Theorem III, and one will have equation (16). Let there then be surrogated in equation (4) of Theorem II, in place of $\dfrac{dx\sqrt{2}}{\sqrt{1+x^4}}$, its value $\dfrac{2\,du}{\sqrt{1-u^4}}$, which comes from the supposition made of x (as equation (6) of Theorem III multiplied by 2 shows), and one will have equation (17).

Which was to be demonstrated.

Corollary (fig. 29). — Calling z the chord CS of the lemniscate, and u the other chord CI, and thereafter integrating equation (17), one obtains Arc. $SL = 2$ arc. CI.

Provided that u and z have the value that is deduced from equation (16).

Problem III (fig. 33). — Let the quadrant of the lemniscate be bisected at M; and let the direct arc CB, less than half the quadrant, be given: to find the intermediate arc MI equal to the given direct arc CB.

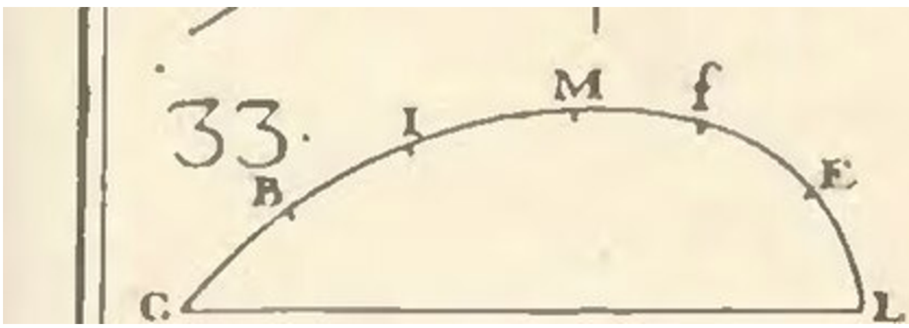

Figure 4: Fig. 33.

Solution. — I. Let there be taken, by means of Theorem VII and its corollary, the inverse arc EL double the given direct arc CB. II. Let there be taken, by means of Theorem V and its Corollary I, the direct arc CI subduple of the direct arc CE. It is evident that the sum of the arcs $CB + CI$, being equal to half the sum of the arcs $CE + EL$, will consequently be equal to half the quadrant; therefore

$$\text{arc.}\, CB + \text{arc.}\, CI = \text{arc.}\, CM,$$

and taking away from both members of this equation the common arc CI, there remains arc. $CB =$ arc. MI.

Which was to be demonstrated.

Corollary (fig. 33). — Let there be taken, by means of Theorem III of Sketch I, the inverse arc FL equal to the direct arc CI, and one will have

$$\text{arc.}\, Mf = \text{arc.}\, MI = \text{arc.}\, CB.$$

Theorem VIII. — Let the two equations (18) and (19) written below be given; I say that, the first of them being posited, the other also holds

$$\frac{\sqrt{1-t^4}}{t\sqrt{2}} = \frac{\sqrt{1-z}}{\sqrt{1+z}}. \tag{18}$$

$$\frac{-dz}{\sqrt{1-z^4}} = -\frac{2\,dt}{\sqrt{1-t^4}}. \tag{19}$$

Proof. — In equation (3) of Theorem II let the sign $\mp$ represent the negative sign, and in place of x let there be put its value $\dfrac{\sqrt{1-t^4}}{t\sqrt{2}}$ expressed in equation (7) of Theorem IV, and one will have equation (18). Let there then be substituted in equation (4) of Theorem II, in place of $\dfrac{dx\sqrt{2}}{\sqrt{1+x^4}}$, its value $-\dfrac{2\,dt}{\sqrt{1-t^4}}$, which results from the supposition made of x (as equation (8) of Theorem IV multiplied by 2 shows), and one will arrive at equation (19).

Corollary (fig. 29). — Naming z the chord CS of the lemniscate, and (t) the other chord CH, attributing to z and to t their due values deduced from equation (18), and integrating equation (19), one discovers the inverse arc SL equal to double the inverse arc HL.

Scholium III. — From Theorems V, VI, VII, and VIII, and their corollaries, readers will be able to deduce ways of bisecting the quadrant of the lemniscate entirely uniform with the manner of doing so discovered by me in Sketch I, and this will serve as proof of the correctness and fruitfulness of my method.

Problem IV. — Given that the quadrant of the lemniscate is cut into a given number of equal parts, to subdivide each of the said parts into two equally equal portions.

Solution. — For greater clarity and brevity let those direct arcs of the lemniscate that contain an odd number of its equal parts be called odd direct arcs. By virtue of Corollary I of Theorem V let the direct arcs subduple of all the odd arcs be taken; then, by Corollary I of Theorem III of Sketch I, let the inverse arcs equal to all these subduple direct arcs be found, and the intent will be obtained.

Which was to be found.

Scholium IV. — From Theorem VIII readers will be able to deduce another, similar way of resolving this problem, which I omit for brevity.

Corollary. — Since in Sketch I I have taught the way of cutting the quadrant of the lemniscate into two equal parts, and in Problems I and II of the present paper I have found the way of cutting the same quadrant into three and five equal parts, it follows, by virtue of the solution of this problem, that the quadrant of the lemniscate can be divided algebraically into as many equal parts as are contained in these three formulas, namely: 2×2^m; 3×2^m; 5×2^m, in which the exponent m signifies any positive whole number.

And this is a new and singular property of my curve.