XXXII. Method for measuring the lemniscate

Sketch I (*)

(*) Giornale de' letterati d'Italia, vol. XXIX, p. 258.

The two eminent geometers Signor Giacomo and Signor Giovanni, the brothers Bernoulli, have made the lemniscate famous, using its arcs to construct the paracentric isochrone, as may be seen in the Acta of Leipzig of the year 1694. It is evident that, by measuring the lemniscate by means of some other curve simpler than it, one obtains a more perfect construction not only of the paracentric isochrone, but also of the other infinite curves which, in order to be constructed, may depend on the lemniscate; and therefore I flatter myself that the measures of this curve discovered by me, which I shall set forth successively in two sketches, will not displease those versed in the matter.

Assumptions known to those versed in the infinitesimal calculus. — Let $CQACFC$ (fig. 24) be the lemniscate, whose semi-axis $CA = a$; it is known that, taking the origin of the abscissae (x) at the centre C, and calling (y) the ordinates normal to the axis, the nature of the lemniscate is expressed by this equation $x^2 + y^2 = a\sqrt{x^2 - y^2}$.

It is also known that if one calls z the indeterminate chord $CQ = \sqrt{x^2 + y^2}$, one has the direct arc $CQ = \int \frac{a^2\,dz}{\sqrt{a^4 - z^4}}$, and the inverse arc

$$QA = \text{arc.}\, CA - \text{arc.}\, CQ = \int \frac{a^2\,dz}{\sqrt{a^4 - z^4}}.$$

Let $ADFNA$ (fig. 24) be the ellipse, whose minor semi-axis $CF = a$, and major semi-axis $CD = a\sqrt{2}$; call z the indeterminate abscissa CH, which has its origin at the centre C of the ellipse and is equal to the chord CQ of the lemniscate, and let the ordinate HI be drawn parallel to the major axis; it is already known that the direct arc DI of this ellipse has for its expression

$$\int dz\,\frac{\sqrt{a^2 + z^2}}{\sqrt{a^2 - z^2}},$$

and that the inverse arc

$$IF = \text{arc.}\, DF - \text{arc.}\, DI = \int -dz\,\frac{\sqrt{a^2 + z^2}}{\sqrt{a^2 - z^2}}.$$

Let there be, finally, the equilateral hyperbola LMP (fig. 25), whose semi-axis $SM = a$. Let t be named the indeterminate ordinate SO, which, starting from the centre S, cuts the arc MO; it is known that this same arc is expressed thus

$$\int \frac{t^2\,dt}{\sqrt{t^4 - a^4}}.$$

Theorem I. — Let the two equations (1) and (2) written below be given. I say that, the first of them being posited, the other also holds

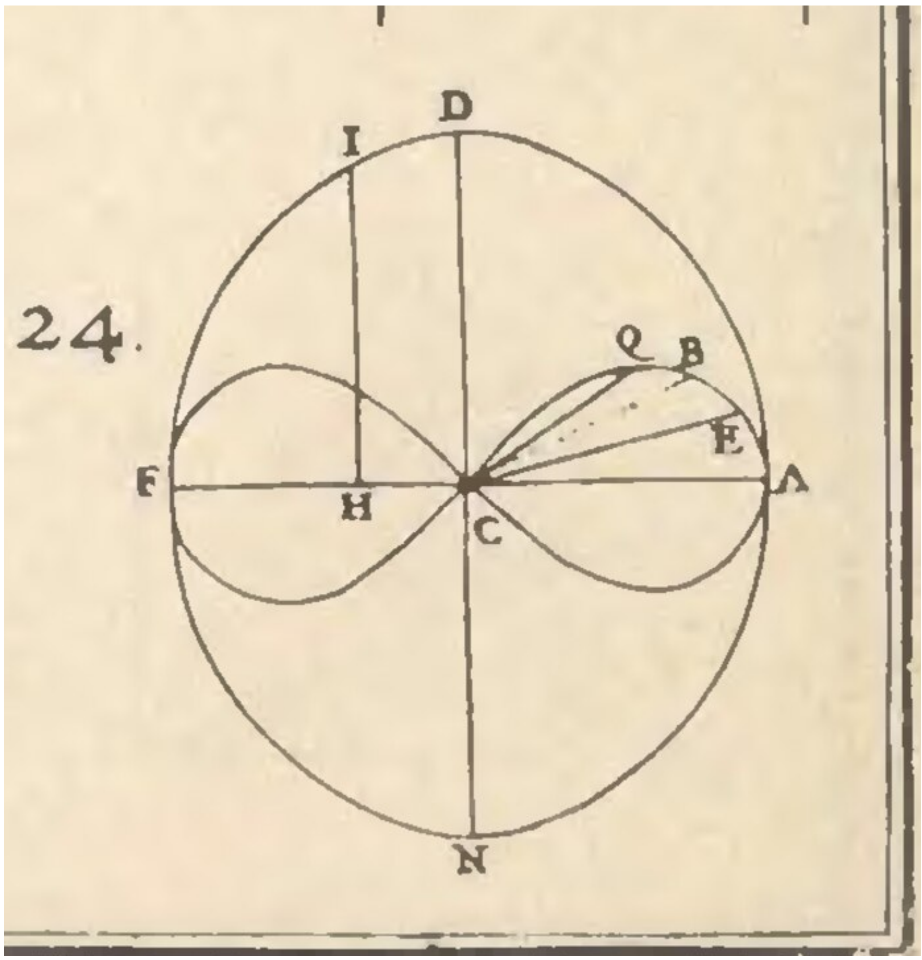

Figure 1: Fig. 24.

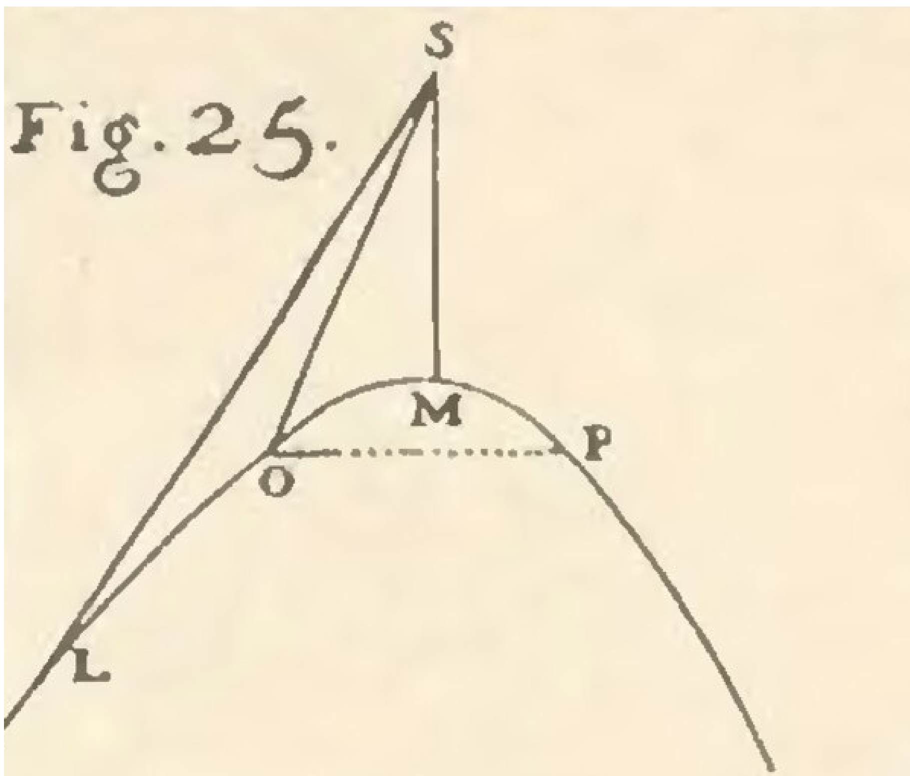

Figure 2: Fig. 25.

$$t = a\,\frac{\sqrt{a^2+z^2}}{\sqrt{a^2-z^2}}. \tag{1}$$

$$\int \frac{a^2\,dz}{\sqrt{a^4-z^4}} = \int dz\,\frac{\sqrt{a^2+zz}}{\sqrt{a^2-az}} + \int \frac{t^2\,dt}{\sqrt{t^4-a^4}} - \frac{zt}{a}. \tag{2}$$

The truth of this theorem is demonstrated by differentiating, and substituting in place of t and dt their values in z and dz drawn from equation (1).

Corollary (fig. 24 and 25). — If in the lemniscate the chord $CQ = z$, and in the ellipse the abscissa CH is likewise $= z$, and in the equilateral hyperbola LMP the central ordinate $SO = t$; assigning to t its value expressed in equation (1), and putting in equation (2) the arcs of the curves instead of their expressions already declared in the assumptions above, one obtains

$$\text{Arc.}\,CQ = \text{arc.}\,DI + \text{arc.}\,MO = \frac{zt}{a}. \tag{3}$$

Scholium I. — Supposing $z = 0$, the direct arcs CQ of the lemniscate and DI of the ellipse are equal to zero; the hyperbolic arc MO is likewise null, since by virtue of equation (1) the vanishing of z makes $SO\,(t) = a = SM$; the rectilinear expression is also annihilated, and all this is a sure indication that equation (3) is complete and needs no addition or subtraction of any constant quantity. But putting $z = a$, then the direct arc of the lemniscate becomes equal to the arc CQA, and the elliptic arc DI to the arc DIF; but by equation (1) the central ordinate $SO\,(t)$ of the hyperbola becomes in this case infinite, rendering the hyperbolic arc MO infinite as well, and therefore it seems impossible to arrive by this route at knowing the value of the quadrant of the lemniscate expressed in finite quantities, and consequently the measure of this curve cannot yet be said to be perfectly discovered.

Theorem II. — Let the two equations (4) and (5) written below be given; I say that, the first of them being posited, the other also holds

$$r = \frac{a^2}{z}. \tag{4}$$

$$\int -\frac{a^2\,dz}{\sqrt{a^4-z^4}} = \int -dz\,\frac{\sqrt{a^2+z^2}}{\sqrt{a^2-z^2}} + \int \frac{r^2\,dr}{\sqrt{r^4-a^4}} - \frac{1}{z}\sqrt{a^4-z^4}. \tag{5}$$

The differentiation and the substitution of the values of r and dr in z and dz deduced from equation (4) demonstrate this theorem.

Corollary I (fig. 24 and 25). — Let there be made, as above, in the lemniscate the chord $CQ = z$, in the ellipse the abscissa CH likewise $= z$, and in the hyperbola the central ordinate $SL = r$, attributing to r its value noted in equation (4), and substituting in equation (5) the arcs of the curves in place of their expressions, as was done in the corollary of Theorem I, one finds again

$$\text{Arc.}\,QA = \text{arc.}\,IF + \text{arc.}\,ML - \frac{1}{z}\sqrt{a^4-z^4}. \tag{6}$$

Scholium II. — The supposition $z = a$ makes null the inverse arcs QA of the lemniscate and IF of the ellipse; it also annuls the rectilinear expression and the hyperbolic arc ML, seeing that, by force of equation (4), the central ordinate of the hyperbola $SL\,(r)$ becomes equal to the semi-axis $SM\,(a)$; hence equation (5) is complete. But the supposition $z = 0$ prolongs to infinity the central ordinate of the hyperbola, and its arc, at the same time as it makes the inverse arcs of the lemniscate and of the ellipse equal respectively to the quadrants ACQ, FID; and here the same difficulty reappears that was set out at the end of Scholium I; but this difficulty will be overcome by the corollary that follows. Meanwhile let it be noted: first, that by drawing in the hyperbola the OP parallel to the second axis, the arc LMP is equal to the two arcs MO, ML. Secondly, that assigning to t its value expressed in equation (1), this complex quantity $\frac{zt}{a} + \frac{1}{z}\sqrt{a^4 - z^4}$ is equivalent to this other, simpler quantity $\frac{at}{z}$, as will be verified by the calculation. **Corollary II** (fig. 24 and 25). — Adding the two equations (3) and (6), and making use of the two remarks noted at the end of the preceding scholium, one discovers $\text{arc.}\,CQA = \text{arc.}\,DIF + \text{arc.}\,LMP - \frac{at}{z}$; and finally, by multiplying this last equation by 4, one finds again that the entire perimeter of the lemniscate is equal to the entire perimeter of the circumscribed ellipse $ADFNA$, plus four times the arc LMP of the equilateral hyperbola minus $\frac{4at}{z}$ — which is a new and excellent property of these curves.

Theorem III. — Let the two equations (7) and (8) written below be given; I say that, the first of them being posited, the other also holds

$$u = a\,\frac{\sqrt{a^2 - z^2}}{\sqrt{a^2 + z^2}}. \tag{7}$$

$$\int \frac{a^2\,dz}{\sqrt{a^4 - z^4}} = \int -\frac{a^2\,du}{\sqrt{a^4 - u^4}}. \tag{8}$$

This theorem is demonstrated, like the other two that precede it, by differentiating and making the due substitutions drawn from equation (7).

Corollary I (fig. 24). — Let there be taken in the lemniscate the chord $CQ = z$, and the other chord $CE = u$; let there be attributed to u its value expressed in equation (7); let the direct and inverse arcs of the curve be substituted in equation (8) in place of their expressions, and one will find $\text{arc.}\,CQ = \text{arc.}\,EA$.

Scholium III (fig. 24). — If $z = 0$ the direct arc CQ is null, and by virtue of equation (7) the vanishing of z makes $CE\,(u) = a$, so that the inverse arc EA is null as well; which makes known that the equation noted at the end of the preceding corollary is complete. This notable property of the lemniscate was discovered by me, and set forth in a different manner in my method for rectifying the difference of the arcs in infinite kinds of non-rectifiable parabolas.

Corollary II (fig. 24). — Finding, as I have taught above to do, the inverse arc EA equal to the direct arc CQ, this same arc CQ may be measured by means of Corollary I of Theorem II; and conversely, finding the direct arc CQ equal to the inverse arc EA, this same arc EA may be measured by means of the corollary of Theorem I. **Corollary III. — Problem.** — To bisect the quadrant of the lemniscate (fig. 24).

If the chord $CQ\,(z)$ of the lemniscate is equal to the other chord $CE\,(u)$ of it, and if at the same time u has its value noted in equation (7), then $z = u = a\,\dfrac{\sqrt{a^2 - z^2}}{\sqrt{a^2 + z^2}}$, whence there results, the due operations being performed,

$$z = u = \sqrt{a^2\sqrt{2} - a^2}$$

and therefore, attributing to z this last value, the two points of the lemniscate Q and E will coincide in B, in such a way that the direct arc CB will be equal to the inverse arc BA, and the chord $CB = u = z$ will bisect the quadrant of the lemniscate. Which was to be found.