Jac. B. Construction of the Curve of uniform Approach & Recession

by means of the rectification of a certain algebraic Curve: an Addendum to the recent solution of the Month of June

There are chiefly three modes of constructing mechanical, or transcendental, curves. The first, but little suited to practice, is done through the quadratures of curvilinear spaces. Better is that which is set up through the rectifications of algebraic curves; for curves can be rectified in practice more accurately and expeditiously, by means of a thread or a little chain wrapped around them, than spaces can be squared. In the same rank I hold those constructions which are carried out without any rectification and quadrature, by the mere description of some mechanical curve whose points, though not all, yet infinitely many and as close together as one pleases, can be found geometrically, such as the Logarithmic curve is wont to be, and any others there may be of that kind. But the best mode, wherever it can be had, is that which is carried out by means of some curve which nature herself, without art, produces by a certain most swift and, as it were, instantaneous motion at the beck of the Geometer; whereas the preceding modes require curves whose

delineation, whether by continuous motion or by the finding of several points, being set up by the Craftsman, commonly turns out either too slow or too much impeded: thus those constructions of Problems which suppose the quadrature of the Hyperbola or the description of the Logarithmic curve I judge, other things being equal, to be inferior to those which are carried out by means of the Catenary, that is, of the curve which a suspended chain assumes of its own accord sooner than you could set the first hand to describing the others.

The Curve of uniform Approach and Recession, for the sole recommendation of the Gentleman who proposed it, seems to deserve that we should take pains to furnish it with constructions according to all three modes. The construction of the first mode, by quadratures, which I passed over in the recent solution for the reason already stated, can be set up by taking, in *Fig.* 3, *Tab.* VI, the straight line EN equal, not, as there, to Au itself, but to iu; and again the rectangle AO made equal to the curvilinear area AEN; for its side AP will be adequate to the recent curve AQ, so that, taking $A\alpha$ as the third proportional to AB and the now-found AP, the point α is had on the desired Curve. The construction of the third mode, which would be done by means of the Elastic line AQ, which alone I touched upon there, would without doubt be the best, if nature had anywhere made tensions simply proportional to the stretching forces: but since perhaps no Spring is found which observes this law of tension precisely; nor, if one were found, would that be certain; therefore it is not safe enough to trust to it either; and it is preferable to have recourse to the second mode of construction, and to seek an algebraic Curve by whose rectification we may attain our aim. Such a one, as if by wish, presents itself: a Curve of four dimensions which is expressed by this equation $xx + yy = a\sqrt{xx - yy}$, and which, set about the axis BG $(2a)$, refers in form to a reclining figure of the numeral eight ∞, or to a band folded into a knot, that is, into a lemniscus, *d'un nœud de ruban* as the French call it. For of this curve both the one half is [equal to] the Elastic curve $AR\varphi$, and, above the node A, over the indeterminate interval AE cut off, the smaller portion of the half [equals] the smaller AQ, the larger [equals] the larger Elastic portion φRQ. Whence too the reason is established for employing it in the construction of the proposed Problem.

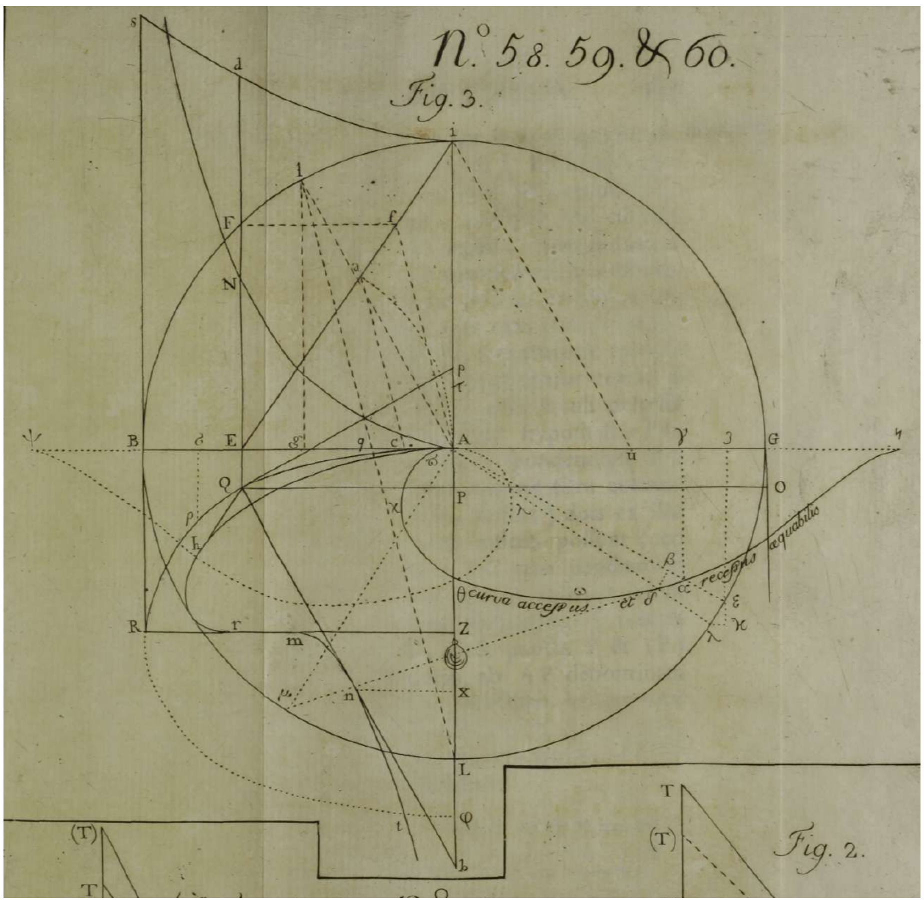

Figure 1: Fig. 3.

For the rest, just as infinitely many mechanical lines depend on the rectification of the circular or parabolic curve, or on the logarithmic description; so too, besides the curve of approach and recession of infinitely many others, the construction of the Elastic curve itself (as will appear

below) is in part derived from the rectification of the said algebraic curve of four dimensions. And I would dare to assert that, in the matter of the construction of mechanical curves, this one immediately follows the preceding among the rest; so much so that a construction which succeeds by none of the earlier ones is to be attempted next either by the rectification of this Curve, or of a Hyperbolic or an Elliptic line, or of two at once. The reason for this is that, after the differential formulas $aa\,dz : \sqrt{aa - zz}$, $zz\,dz : \sqrt{aa - zz}$, $aa\,dz : \sqrt{zz - aa}$, $zz\,dz : \sqrt{zz - aa}$, which are integrated by means of the quadrature of the Circle and the Hyperbola, the simplest by far seem to be these expressions, $zz\,dz : \sqrt{z^4 - a^4}$, $aa\,dz : \sqrt{z^4 - a^4}$, $aa\,dz : \sqrt{a^4 - z^4}$, $zz\,dz : \sqrt{a^4 - z^4}$, and the like; of which the first is integrated by means of a Hyperbolic line, the second and third by the rectification of our lemniscate curve, the fourth by that of the same and of an Elliptic curve. For indeed if the indeterminate z be applied to an equilateral Hyperbola (whose transverse axis is $2a$) from its centre, the arc intercepted between the vertex and the applied line will satisfy the first formula. And if the same z (or the third proportional to z and a, where z is greater than a) be subtended from the node of the lemniscate Curve, the subtended arc will serve for the two middle ones. And if, finally, from the centre of an Ellipse (whose minor axis is $2a$, the major $2a\sqrt{2}$) the same z be cut off on the minor axis and through the point of section a straight line be drawn parallel to the major, the elliptic portion intercepted between the parallels will designate it, the corresponding portion of the lemniscate Curve being cut away, the integral of the fourth formula $zz\,dz : \sqrt{a^4 - z^4}$. Whence, since the same line in the cited figure, $AE = z$ being given, denotes the element of the applied line AP or EQ in the Elastic curve, it is plain that this applied line equals the difference of two portions of the Elliptic and the Lemniscate curves, and therefore it is manifest how the Elastic curve can be constructed by the rectification of both. And since the portion of the Lemniscate, as I noted above, is adequate to the Elastic portion AQ, it is clear also that the Elliptic curve itself is adequate to the aggregate $AQ + AP$; which stands as a not inelegant property of these Curves.