Mathematical Analysis. — On integrals of total differentials. Note by M. H. Poincaré, presented by M. Hermite.

«The recent discovery of M. Picard on total differentials of the first kind has opened to analysts an entirely new path, where they will undoubtedly encounter many important propositions. It will therefore perhaps not be without interest to point out here certain partial results which, although very easy to prove, may be useful to geometers who concern themselves with this question.

»I sought first to determine which surfaces of the fourth order possess integrals of the first kind. I found that all these surfaces can be reduced, by a linear change of variables, either to the ruled surface

$$x^2(az^2 + 2b - z + c) + 2xy(a'z^2 + 2b'z + c') \\ + y^2(a''z^2 + 2b''z + c'') = 0,$$

[1] which admits the integral

$$\int \frac{y\,dz}{x(az^2 + 2bz + c) + y(a'z^2 + 2b'z + c')},$$

or to the surface of revolution

$$(x^2 + y^2)^2 + 2(x^2 + y^2)Z_2 + Z_4 = 0$$

(Z_2 and Z_4 denoting two polygons[2] of degree 2 and 4 in z), which admits the integral

$$\int \frac{dz}{x^2 + y^2 + Z_2}.$$

»It is, moreover, easy to see that all ruled surfaces and all surfaces of revolution admit integrals of the first kind, unless, of course, they are unicursal.

»If a surface admits an integral of the first kind u, reducible to elliptic integrals, the curves $u =$ const. are algebraic.

»If one can trace on a surface a unicursal curve, and if u is an arbitrary integral of the first kind of this surface, the value of u will be the same all along the curve.

»Likewise, if the surface admits a conical point of the second order, whose tangent cone is indecomposable, the value of u at this conical point will be *determinate*.

»If one can trace on a surface two series of unicursal curves, it will have no integral of the first kind; if, on a non-unicursal surface, one can trace a series of unicursal curves, in such a way that through each point of the surface there

[1] *Editorial note:* Printed $2b - z + c$ — an apparent misprint for $2bz$, matching $2b'z$ and $2b''z$, and the denominator of the integral immediately following. Reproduced here as printed.

[2] *Editorial note:* Printed “polygones” — an apparent compositor’s slip for “polynômes”. Reproduced here as printed.

passes, in general, only a single one of these curves, it will have integrals of the first kind.

»Suppose that a surface is generated by the elimination of two parameters a and b, between the three equations

$$\varphi(x, y, z, a, b) = 0,$$
$$\varphi_1(x, y, z, a, b) = 0,$$
$$\psi(a, b) = 0.$$

»If the three polynomials φ, φ_1 and ψ are *the most general of their degrees*, the relation $\psi = 0$ is of genus greater than 0; to a point of the surface corresponds a single system of values of the parameters and, consequently, the surface admits integrals of the first kind.

»Finally, Abel's theorem applies to integrals of total differentials.

»Let $M_1, M_2, \ldots, M_q$ be the points of intersection of the surface with the curve

$$\frac{\alpha}{\lambda} = \frac{\beta}{\mu} = \frac{\gamma}{\nu},$$

α, β, γ being polynomials in x, y, z and λ, μ, ν constants. Let $u_1, u_2, \ldots, u_q$ be the values of a certain integral of the first kind u at these different points.

»Let $M_1', M_2', \ldots, M_q'$ be the points of intersection of the surface with the curve

$$\frac{\alpha}{\lambda'} = \frac{\beta}{\mu'} = \frac{\gamma}{\nu'},$$

λ', μ' and ν' being new constants. Let $u_1', u_2', \ldots, u_q'$ be the values of the integral u at the points $M_1', M_2', \ldots, M_q'$.

»We will have

$$u_1 + u_2 + \ldots + u_q = u_1' + u_2' + \ldots + u_q',$$

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