On the so-called Dirichlet principle.

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In his lectures on the forces which act according to Newton's law, *Lejeune Dirichlet* made use, in order to establish a main theorem of potential theory, of a peculiar mode of inference which has since been applied in many ways by other mathematicians as well, in particular by *Riemann*, and which has received the name »Dirichlet principle«.

Concerning the admissibility of this »principle« various doubts have since made themselves felt, which, as I shall show in what follows, are thoroughly well founded. Before I enter upon them, however, since Dirichlet has published nothing on the subject in question, I wish to communicate the following passage from a lecture delivered by Dirichlet in the summer semester of 1856, which lies before me in a careful transcript prepared by Herr *Dedekind*, and from which it emerges with full definiteness what Dirichlet meant and how he sought to justify his method of proof:

»If any finite surface whatever is given, then one can always, but only in one way, cover it with mass in such a way that the potential at every point of the surface has an arbitrarily prescribed value (varying according to continuity).«

»For the proof we premise the following theorem:

»If any finite connected space t whatever is given, then there always exists one, but only one single function w which, together with its first derivatives, varies continuously everywhere in t, which on the boundary of t everywhere assumes arbitrarily prescribed (continuously variable) values, and which within t satisfies the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

everywhere. — This theorem is properly identical with another one from the theory of heat, which there appears immediately evident to everyone, namely that, if the boundary of t is kept constant at an everywhere arbitrarily prescribed temperature, there always exists one, but also only one, distribution of temperature in the interior at which equilibrium subsists; or that, as one may also say, if the original temperature in the interior was an arbitrary one, it approaches a final state at which equilibrium would subsist.«

»We prove the theorem by starting from a purely mathematical evidence. It is in fact clear that among all functions u which, together with their first derivatives, vary continuously everywhere in t and assume the prescribed values on the boundary of t, there must be one (or several) for which the integral extended through the whole space t

$$U = \int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} dt$$

receives its *smallest* value. We will denote such a function precisely by u and the minimum of the integral by U. Now let u' be any other function which fulfils the same boundary and continuity conditions as u, and U' the corresponding value of the integral; then $U'-U$ can never be negative. If we now set

$$u + hw = u',$$

where h denotes an indeterminate constant factor, then w is a function which fulfils the same continuity conditions as u and u' and becomes equal to zero everywhere on the boundary of t, but is otherwise entirely arbitrary. Then one easily finds

$$U' - U = 2hM + h^2 N,$$

where

$$M = \int \left\{ \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \right\} dt,$$

$$N = \int \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right\} dt$$

By integration by parts, having regard to the boundary and continuity conditions for w and the continuity conditions for the first derivatives of u, one easily finds

$$M = -\int w \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right\} dt.$$

Now since $U'-U$ can never be negative for any h, however small it may be, and N is a finite positive quantity, it follows that M is necessarily equal to zero; and since this must be the case however w may be constituted, we must conclude that everywhere within t (at most with the exception of isolated surfaces, lines, points)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

must hold; for if this trinomial were different from zero in a solid space, one would need only to give w everywhere the same sign as that trinomial in order to obtain a value different from zero for M.«

»There exists therefore in any case such a function u, which fulfils the stated boundary and continuity conditions and satisfies the partial differential equation. But there also exists only *one single* such function u. For if $u' = u + w$ is any other function which fulfils the same boundary and continuity conditions, then the corresponding integral is

$$U' = U + \int \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right\} dt,$$

and consequently U is really an *absolute minimum*; and if perchance $U' = U$ should hold, then, as is easily seen, in the whole space t

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = 0$$

would have to hold. From this it would follow that w is constant; and since it is continuous and on the boundary of t is zero, it must be zero everywhere, that is, u' must be identical with u.«

»It will be shown later that this u, which fulfils the boundary and continuity conditions, and of which we know that it contradicts the partial differential equation at most in isolated points, lines or surfaces within t, must satisfy it *everywhere*, that is, at every point, so that such exceptional places are impossible.«

Under the assumption that for the space t there exists at all a function satisfying the prescribed boundary and continuity conditions, the Dirichlet procedure set forth above is accordingly to serve to demonstrate the existence of a function of x, y, z which satisfies the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and at the same time the stated subsidiary conditions. The latter is easy to prove in every determinate case; the other, however, holds only in the case where it can be shown that there exists a function u for which the value of the expression

$$\int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} dt,$$

is an (absolute) minimum. This, however, can by no means be inferred from the assumptions made by Dirichlet; it can only be asserted that for the expression in question there exists a determinate *lower bound* which it can approach arbitrarily closely without having actually to attain it. Accordingly Dirichlet's mode of inference proves to be untenable.

Finally I will illustrate the foregoing by a simple example, through which the inadmissibility of the Dirichlet mode of inference is evidently demonstrated.

Let $\varphi(x)$ denote, namely, a real single-valued function of the real variable x of such a nature that, first, $\varphi(x)$ and $\frac{d\varphi(x)}{dx}$ are continuous functions of x in the interval $(-1 \cdots +1)$, and that, secondly, $\varphi(x)$ has at the boundary -1 of the interval the prescribed value a, and at the boundary $+1$ the prescribed value b. Here the two constants a and b are to be two quantities different from one another. If now the Dirichlet mode of inference were admissible, then there would have to be found among the functions $\varphi(x)$ under consideration a special function for which the value of the integral

$$J = \int_{-1}^{+1} \left(x \frac{d\varphi(x)}{dx} \right)^2 dx$$

is equal to the lower bound of all those values which this integral can assume for the various functions $\varphi(x)$ belonging to the totality under consideration.

The lower bound mentioned is, however, in the present case necessarily equal to zero. For if one sets, for example,

$$\varphi(x) = \frac{a+b}{2} + \frac{b-a}{2} \frac{\operatorname{arctg} \frac{x}{\varepsilon}}{\operatorname{arctg} \frac{1}{\varepsilon}},$$

where ε denotes an arbitrarily assumable *positive* quantity, then this function fulfils the first two conditions; and since

$$J < \int_{-1}^{+1} (x^2 + \varepsilon^2) \left(\frac{d\varphi(x)}{dx} \right)^2 \, dx$$

and

$$\frac{d\varphi(x)}{dx} = \frac{b-a}{2 \operatorname{arctg} \frac{1}{\varepsilon}} \cdot \frac{\varepsilon}{x^2 + \varepsilon^2},$$

then for this special function

$$J < \varepsilon \frac{(b-a)^2}{\left(2 \operatorname{arctg} \frac{1}{\varepsilon}\right)^2} \int_{-1}^{+1} \frac{\varepsilon \, dx}{x^2 + \varepsilon^2}$$

and thus

$$J < \frac{\varepsilon}{2} \frac{(b-a)^2}{\operatorname{arctg} \frac{1}{\varepsilon}}.$$

From this it is clear, since one can assume ε arbitrarily small, that the lower bound of the value of J is equal to zero, for J cannot assume negative values at all.

This bound, however, the value of J cannot attain, however one may choose the function $\varphi(x)$ in accordance with the two conditions above. For since $\varphi(x)$ and $\frac{d\varphi(x)}{dx}$ are to be continuous functions of x, it would be requisite for this that

$$\frac{d\varphi(x)}{dx}$$

should vanish for every value of x belonging to the interval $(-1 \cdots + 1)$, and therefore that $\varphi(x)$ should be a constant. This, however, is incompatible with the assumption that a and b are different from one another.

The Dirichlet mode of inference therefore evidently leads, in the case considered, to a false result.